Module7: Strength and Failure Theories

Learning Unit-1: M7.1

M7.1 Strength of unidirectional composites and laminates

M7.1.1 Introduction Failure:

Every material has certain strength, expressed in terms of stress or strain, beyond which it fractures or fails to carry the load. Failure Criterion: A criterion used to hypothesize the failure. Failure Theory: A Theory behind a failure criterion.

Need of Failure Theories:

(a) To design structural components and calculate margin of safety.
(b) To guide in materials development.
(c) To determine weak and strong directions.

Failure Theories for Isotropic Materials: Strength and stiffness are independent of the direction. Failure in metallic materials is characterized by Yield Strength.

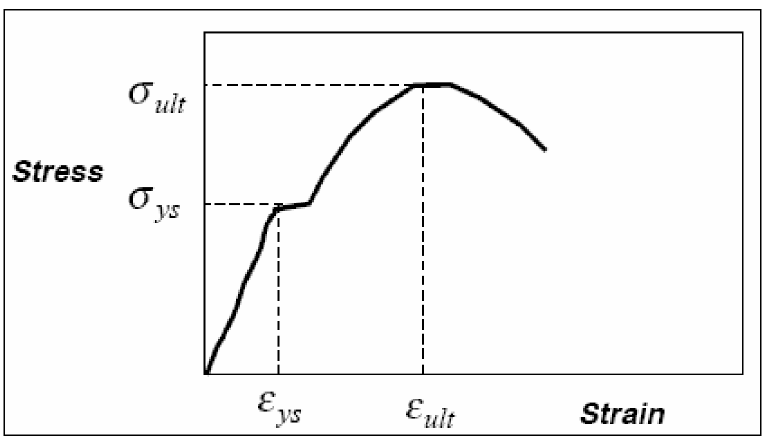


Figure: Stress-strain curve

Strain Theories:

- Maximum principal stress theory.
- Maximum principal strain theory.
- Quadratic or Distortional Energy Theory.

Macromechanical Failure Theories in Composite Materials:

- Maximum Stress Theory
- Maximum Strain Theory
- Tsai-Hill Theory (Deviatoric strain energy theory)

- Tsai-Wu Theory (Interactive tensor polynomial theory)

Application of Failure Theory

First step is to calculate the stresses/strains in the material principal directions. This can be done by transformation of stresses from the global coordinates to local material coordinates of the ply.

Ply Stresses:

$$\{\sigma\}_{x-y} = [T_\sigma]\{\sigma\}_{1-2} \quad \text{or} \quad \{\sigma\}_{1-2} = [T_\sigma]^{-1}\{\sigma\}_{x-y}$$

Ply strains:

$$\{\varepsilon\}_{1-2} = [Q]_{1-2}\{\sigma\}_{1-2}$$

Now apply the failure criteria in the material coordinate system.

We can use the constitutive equations (stress-strain relationships for an individual lamina/ply or hence laminate) to calculate the stresses in each ply when we know the values of the loads acting on the laminate. By comparing these stresses with a corresponding limiting value we can decide whether, or not, the laminate will fail when subjected to the service loads.

There are several ways to define failure. The obvious one is when we have complete separation, or fracture; clearly, then, the component can no longer support the loads acting on it. However, a more general definition would be 'when the component can no longer fulfill the function for which it was designed'.

Such a definition includes total fracture but could also include excessive deflection as seen when a laminate buckles (basically a **stiffness** rather than a **strength** limit), or even just matrix cracking. The latter could constitute failure for a container because any contents would be able to leak through the matrix cracks in the container's walls.

As for isotropic materials, a failure criterion can be used to predict failure. A large number of such criteria exist, no one criterion being universally satisfactory. We shall start by considering a single ply before moving on to discuss laminates.

The lamina to be a regular array of parallel continuous fibres perfectly bonded to the matrix. We know that, there are five basic modes of failure of such a ply: longitudinal tensile or compressive, transverse tensile or compressive, or shear. Each of these modes would involve detailed failure mechanisms associated with fibre, matrix or interface failure. Some typical strength of PMCs are shown in Table M7.1.1.

We can regard the strengths in the principal material axes (parallel and transverse to the fibres) as the fundamental parameters defining failure. When a ply is loaded at an angle to the fibres, as it is when it is part of a multidirectional laminate, we have to determine the stresses in the principal directions and compare them with the fundamental values.

Material	Longitudinal	Longitudinal compression	Transverse tension	Transverse compression	Shear
Glass-polyester	650-950	600-900	20-25	90-120	45-60
Carbon-epoxy	850-1500	700-1200	35-40	130-190	60-75
Kevlar-epoxy	1100-1250	240-290	20-30	110-140	40-60

Table M7.1.1 Typical strength of unidirectional PMC laminates (ν_f = 0.5) (values in MPa)

M7.1.2 Strength of a Lamina

Strength can be determined by the application of failure criteria, which are usually grouped into three classes: **limit criteria**, the simplest; **interactive criteria** which attempt to allow for the interaction of multiaxial stresses; and **hybrid criteria** which combine selected aspects of limit and interactive methods. In this text we shall only discuss criteria that fit into the first two classes.

7.1.2.1 Limit criteria

(a) Maximum stress criterion

The maximum stress criterion consists of five sub-criteria, or limits, one corresponding to the strength in each of the five fundamental failure modes. If any one of these limits is exceeded, by the corresponding stress expressed in the principal material axes, the material is deemed to have failed.

In mathematical terms we say that failure has occurred if

$$\sigma_1 \geqslant \hat{\sigma}_{1T} \quad \text{or} \quad \sigma_1 \leqslant \hat{\sigma}_{1C} \quad \text{or} \quad \sigma_2 \geqslant \hat{\sigma}_{2T} \quad \text{or} \quad \sigma_2 \leqslant \hat{\sigma}_{2C} \quad \text{or} \quad \tau_{12} \geqslant \hat{\tau}_{12}. \qquad (7.1.1)$$

or,

(Recalling that a compressive stress is taken as negative so, for example, failure would occur if σ_2 =0 MPa and $\hat{\sigma}_{2T} = -150$ MPa).

(b) Maximum strain criterion

The maximum strain criterion merely substitutes strain for stress in the five sub-criteria. We now say that failure has occurred if

$$\varepsilon_1 \geqslant \hat{\varepsilon}_{1T} \quad \text{or} \quad \varepsilon_1 \leqslant \hat{\varepsilon}_{1C} \quad \text{or} \quad \varepsilon_2 \geqslant \hat{\varepsilon}_{2T} \quad \text{or} \quad \varepsilon_2 \leqslant \hat{\varepsilon}_{2C} \quad \text{or} \quad \gamma_{12} \geqslant \hat{\gamma}_{12} \qquad (7.1.2)$$

As when calculating stiffness, it is important that we can deal with the situation in which the fibres are not aligned with the applied stresses. We illustrate this by considering the simple case of a single stress σ_x, inclined at an angle θ to the fibres (Figure M7.1.1). We now use equations $\sigma_{12} = T\sigma_{xy}$ *and* $\bar{\varepsilon}_{12} = T\bar{\varepsilon}_{xy}$ to obtain the stresses in the principal material directions. Putting $\sigma_y = \tau_{xy} = 0$ in those equations we obtain

$$\sigma_1 = \sigma_x \cos^2\theta, \quad \sigma_2 = \sigma_x \sin^2\theta, \quad \tau_{12} = -\sigma_x \sin\theta\cos\theta. \tag{7.1.3}$$

We then apply equations (7.1.1) to determine whether failure has occurred. We are seeking the value of ax to cause failure and we see from equation (7.1.3) that there are three possible results, i.e.

$$\sigma_x = \hat{\sigma}_{1T}/\cos^2\theta \text{ – fibre failure.}$$

or

$$\sigma_x = \hat{\sigma}_{2T}/\sin^2\theta \text{ – transverse failure}$$

or

$$\sigma_x = -\hat{\tau}_{12}/\sin\theta\cos\theta \text{ – shear failure;}$$

clearly the smallest value of σ_x will be the failure stress.

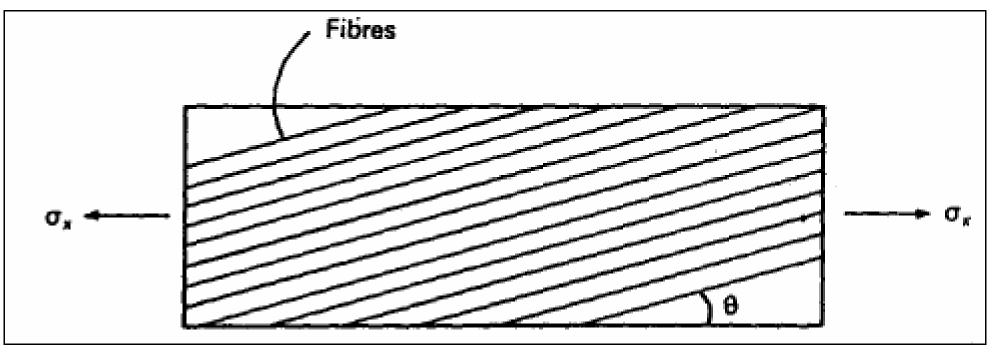


Figure M7.1.1 Uniaxial stresses, σ_x inclined at an angle θ to the fibres of unidirectional composites.

The effect on the value of σ_x at failure as θ is varied as illustrated in Figure M7.1.2. We see that each mode of failure is represented by a separate curve. Fibre failure is most likely when θ is small, transverse (either matrix or interface) failure is when θ approaches 90 °, and shear failure at intermediate angles.

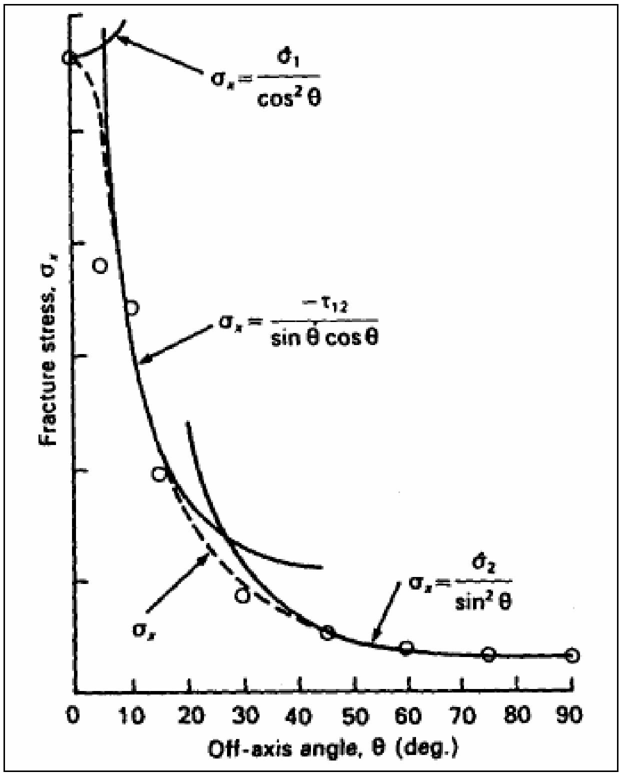


Figure M7.1.2 Variation with θ of σ_x at failure: full lines - maximum stress limit criterion; dotted line - Tsai-Hill interactive criterion.

Example 7.1.1

A tensile specimen of a unidirectional composite is prepared such that the fibres make an angle of 5 ° with the applied load. Determine the stress to cause failure according to (a) the maximum stress criterion, and (b) the maximum strain criterion. The following properties may be used:

$$E_{11} = 76.0,\ E_{22} = 5.5,\ G_{12} = 2.35\,GPa;\ \nu_{12} = 0.33;$$
$$\hat{\sigma}_{1T} = 1250,\ \hat{\sigma}_{2T} = 30,\ \hat{\tau}_{12} = 50,$$
$$\hat{\sigma}_{1C} = 1000,\ \hat{\sigma}_{2C} = 100\ MPa.$$

(a) We first calculate the stresses in the principal material directions as described above, i.e.

$$\sigma_1 = \sigma_x \cos^2\theta, \quad \sigma_2 = \sigma_x \sin^2\theta, \quad \tau_{12} = -\sigma_x \sin\theta\cos\theta$$

where, here, θ = 5°.

Our conditions for failure become (there are no compressive stresses in this instance)

$$\sigma_1 = \sigma_x \cos^2\theta \geqslant \hat{\sigma}_{1T}, \quad \text{or} \quad \sigma_2 = \sigma_x \sin^2\theta \geqslant \hat{\sigma}_{2T},$$

or

$$\tau_{12} = -\sigma_x \sin\theta \cos\theta \geqslant \hat{\tau}_{12}.$$

Taking the equality we get three values of σ_x, the smallest of which is the desired result. Substituting for θ and the given strengths gives

$$\sigma_x = 1250/\cos^2 5 \quad = 1260 \text{ MPa},$$

or

$$\sigma_x = 30/\sin^2 5 \quad = 3949 \text{ MPa},$$

or

$$\sigma_x = 50/\sin 5 \cos 5 = 575 \text{ MPa}.$$

Hence we have a failure stress of 575 MPa, the failure being in a shear mode.

(b) We can determine the direct strains in the principal directions using equation $\varepsilon_1 = \frac{\sigma_1}{E_{11}} - \nu_{21}\frac{\sigma_2}{E_{22}}$, $\varepsilon_2 = -\nu_{12}\frac{\sigma_1}{E_{11}} + \frac{\sigma_2}{E_{22}}$, $\gamma_{12} = \frac{\tau_{12}}{G_{12}}$, noting that they depend on both stresses, i.e.

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}$$

or, using the results from part (a),

$$\varepsilon_1 = \frac{\sigma_1}{E_{11}} - \frac{\nu_{21}}{E_{22}}\sigma_2 = \left(\frac{\cos^2\theta}{E_{11}} - \frac{\nu_{21}\sin^2\theta}{E_{22}}\right)\sigma_x$$

and,

$$\varepsilon_2 = \frac{-\nu_{12}\sigma_1}{E_{11}} + \frac{\sigma_2}{E_{22}} = \left(\frac{-\nu_{12}\cos^2\theta}{E_{11}} + \frac{\sin^2\theta}{E_{22}}\right)\sigma_x.$$

The shear strain is obtained quite simply as,

$$\gamma_{12} = \frac{\tau_{12}}{G_{12}} = \frac{-\sigma_x \sin\theta\cos\theta}{G_{12}}$$

We now need to determine the failure strains from the data given above, assuming the material to have a linear stress-strain relationship, i.e.

$$\hat{\varepsilon}_{1T} = 1250/76 \times 10^3 = 0.0164,$$
$$\hat{\varepsilon}_{2T} = 30/5.5 \times 10^3 = 0.0055,$$
$$\hat{\gamma}_{12} = 50/2.35 \times 10^3 = 0.0218,$$
$$\hat{\varepsilon}_{1C} = -1000/76 \times 10^3 = -0.0132,$$
$$\hat{\varepsilon}_{2C} = -100/5.5 \times 10^3 = -0.0182.$$

The conditions for failure will again give three values for σ_x. Clearly in this problem the values for the shear mode of failure will be the same for both criteria. Also ε_2 will be compressive so we should use $\hat{\varepsilon}_{2C}$ (— 0.0182) as our limiting strain.

Corresponding to ε_1 we get

$$\sigma_x = \hat{\varepsilon}_{1T} \Big/ \left(\frac{\cos^2\theta}{E_{11}} - \frac{\nu_{21}\sin^2\theta}{E_{22}} \right) = 1259 \text{ MPa},$$

and corresponding to ε_2 we get,

$$\sigma_x = \hat{\varepsilon}_{2C} \Big/ \left(\frac{-\nu_{12}\cos^2\theta}{E_{11}} + \frac{\sin^2\theta}{E_{2C}} \right) = 6210 \text{ MPa}.$$

So, finally, the failure stress is seen to be 575 MPa, as given by the maximum stress criterion.

Although they are simple to use, limit criteria do not agree well with experimental data unless the fibre angle is close to 0° or 90°. This is because at intermediate angles there will be a stress field in which both σ_1 and σ_2 can be significant. These stresses will interact and affect the failure load, a situation which is not represented in criteria where a mode of failure is assumed not to be influenced by the presence of other stresses.

Also, the maximum stress and maximum strain criteria will give different predictions when the stress-strain relation is nonlinear. This will certainly be the case for shear deformations and hence the assumption of linearity made in the last example is seen to be invalid. In such cases the maximum strain criterion generally gives better agreement with experiment than does the maximum stress criterion.

7.1.2.2 Interactive criteria

Interactive criteria, as the name suggests, are formulated in such a way that they take account of stress interactions. The objective of this approach is to allow for the fact that failure loads when a multi-axial stress state exists in the material may well differ from those when only a uniaxial stress is acting.

There are many such criteria, of varying complexity, their success in predicting failure often being confined to one fibre/resin combination subjected to a well defined set of stresses (e.g., a tube under internal pressure). The **Tsai-Hill criterion** which has proven to be successful in a wide variety of circumstances is the only method that will be discussed here.

The Tsai-Hill criterion was developed from Hill's anisotropic failure criterion which, in turn, can be traced back to the **Von Mises yield criterion**. In its most general form the Tsai-Hill criterion defines failure as

$$\left(\frac{\sigma_1}{\hat{\sigma}_1}\right)^2 - \frac{\sigma_1\sigma_2}{\hat{\sigma}_1^2} + \left(\frac{\sigma_2}{\hat{\sigma}_2}\right)^2 + \left(\frac{\tau_{12}}{\hat{\tau}_{12}}\right)^2 \geqslant 1. \tag{7.1.4}$$

Because it is usually small compared with the others, the second term $(\sigma_1\sigma_2/\hat{\sigma}_1^2)$ is often neglected. The modified form of the criterion is then

$$\left(\frac{\sigma_1}{\hat{\sigma}_1}\right)^2 + \left(\frac{\sigma_2}{\hat{\sigma}_2}\right)^2 + \left(\frac{\tau_{12}}{\hat{\tau}_{12}}\right)^2 \geqslant 1. \tag{7.1.4(a)}$$

The values of strength used in equation (7.1.4) or (7.1.4a) are chosen to correspond to the nature of σ_1 and -σ_2. So if σ_1 is tensile $\hat{\sigma}_{1T}$ is used, if σ_2 is compressive $\hat{\sigma}_{2C}$ would be used, and so on.

It should be noted that only one criterion has to be satisfied, as opposed to the five sub-criteria of the limit methods. Thus, only one value is obtained for the failure stress. Another point to bear in mind is that the **mode of failure** is not indicated by the method, unlike with the limit criteria. This latter issue has an influence on how we predict the failure of laminates, as we shall see later. For a unidirectional composite subjected to uniaxial stress parallel to a principal direction the Tsai-Hill and maximum stress criteria will give the same failure stress.

As with the limit criteria we are interested in the case of off-axis loading, i.e. stress and fibres not aligned. To illustrate this we again take the simple case of a single stress σ_x acting at θ to the fibres. Substituting the stresses of equation (7.1.3) in equation (7.1.4) gives, at failure,

$$\left(\frac{\sigma_x \cos^2\theta}{\hat{\sigma}_1}\right)^2 - \frac{\sigma_x^2 \cos^2\theta \sin^2\theta}{\hat{\sigma}_1^2} + \left(\frac{\sigma_x \sin^2\theta}{\hat{\sigma}_2}\right)^2 + \left(\frac{\sigma_x \sin\theta\cos\theta}{\hat{\tau}_{12}}\right)^2 = 1. \tag{7.1.5}$$

The way in which σ_x, obtained from the latter equation, varies with θ is shown by the dotted curve in Figure M7.1.2. Note that there is a continuous variation, rather than three separate curves as obtained with the limit criterion.

Example 7.1.2

Repeat Example 7.1.1 using the Tsai-Hill criterion.

We use equation (7.1.5) in conjunction with the appropriate strengths, i.e.

$$\left(\frac{\sigma_x \cos^2\theta}{1250}\right)^2 - \frac{\sigma_x^2 \cos^2\theta \sin^2\theta}{1250^2} + \left(\frac{\sigma_x \sin^2\theta}{30}\right)^2 + \left(\frac{\sigma_x \sin\theta \cos\theta}{50}\right)^2 = 1.$$

Substituting for θ=5° we find at failure
σ_x = 520MPa.

Notice that this does not agree with the prediction of the maximum stress criterion, due to the allowance made here for the interactive effect of the stresses. Examination of the relative magnitudes of the terms in the equation shows that the last is dominant. This indicates that failure would probably be in shear (as predicted by the maximum stress criterion).

We have shown that application of a failure criterion will tell us whether, or not, failure will occur for a given set of stresses. It is clearly also of interest to be able to predict the magnitude of the stress, or stresses, that will cause failure. We can achieve this by simple factoring. If we are dealing with a limit criterion we merely examine the smallest ratio of limit-to-calculated strains and scale accordingly.

Using an interactive criterion is slightly more complicated. Suppose we scale our set of stresses σ_1, σ_2 and τ_{12}, (and the associated Cartesian stresses σ_x, σ_y and τ_{xy}) by the factor **R**. Then equation (7.1.4) becomes

$$\left(\frac{\mathrm{R}\sigma_1}{\hat{\sigma}_1}\right)^2 - \frac{\mathrm{R}\sigma_1 \mathrm{R}\sigma_2}{\hat{\sigma}_1^2} + \left(\frac{\mathrm{R}\sigma_2}{\hat{\sigma}_2}\right)^2 + \left(\frac{\mathrm{R}\tau_{12}}{\hat{\tau}_{12}}\right)^2 = 1 \quad \textbf{at failure,}$$

that is,

$$\left(\frac{\sigma_1}{\hat{\sigma}_1}\right)^2 - \frac{\sigma_1\sigma_2}{\hat{\sigma}_1^2} + \left(\frac{\sigma_2}{\hat{\sigma}_2}\right)^2 + \left(\frac{\tau_{12}}{\hat{\tau}_{12}}\right)^2 = \frac{1}{R^2} \qquad (7.1.6)$$

R is clearly a measure of the available strength in the system. It is often referred to as the **Reserve Factor**. By this factor, we can increase all the stresses by this factor before failure occurs.

7.1.3 Failure of a Laminate

7.1.3.1 Initial failure

Suppose we take a cross-ply laminate (0/90° lay-up) and apply an increasing load in the direction of the 0° fibres. At a relatively low load cracks will be seen in the matrix parallel to the fibres in the 90° plies. These cracks will increase in density until a saturation state is reached. At this point the 90° plies contribute virtually no stiffness to the laminate in the 0° direction, a fact that is shown by the change in slope of the load-extension curve for such a laminate (Figure M7.1.3). The commencement of transverse ply cracking is known as **initial failure or first ply failure**.

Similar behaviour would be seen in an angle-ply laminate ($\pm\theta$ lay-up), with initial failure indicated by cracks parallel to the fibres. These cracks would be caused by interlaminar shear at low values of 'θ' and by the transverse tension at high values of 'θ'.

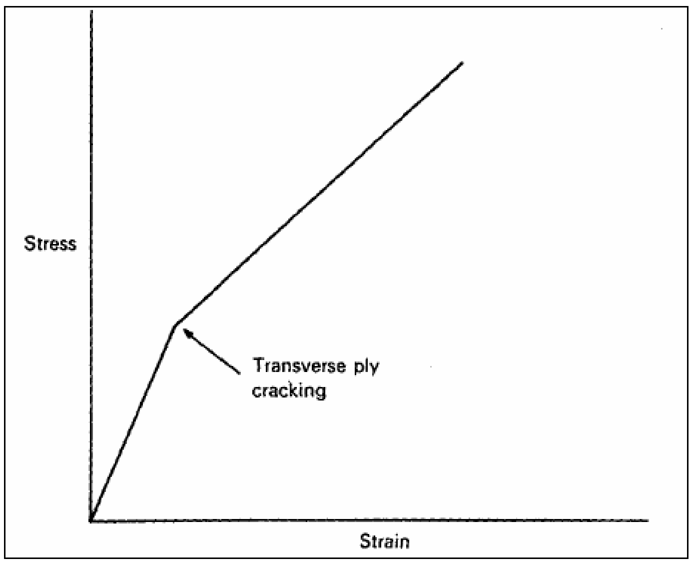


Figure M7.1.3 Change in slope of stress-strain curve for a cross-ply laminate at onset of transverse (90°) ply cracking.

It is possible to predict initial failure of laminate by combining **Classical Laminate Theory** with a failure criterion. Clearly, the choice of criterion is crucial and, as already stated, there are many available, each one often being relevant only to a very specific situation (loading and geometry).

We start with the plate constitutive equations, i.e.

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ \mathbf{k} \end{bmatrix}. \qquad (7.1.7)$$

Solution of equations (7.1.7) will give the plate mid-plane strains (ε°) and plate curvatures (**k**) for a known set of forces (**N**) and moments (**M**).

$$\begin{bmatrix} \varepsilon^0 \\ k \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} = \begin{bmatrix} A^1 & B^1 \\ B^1 & D^1 \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix} \qquad (7.1.8)$$

$$\begin{aligned} A^1 &= A^* + B^*[D^*]^{-1}[B^*]^t, \\ B^1 &= B^* - [D^*]^{-1}, \\ D^1 &= [D^*]^{-1}, \\ A^* &= A^{-1}, \\ B^* &= A^{-1}B, \\ D^* &= D - BA^{-1}B. \end{aligned}$$

The plate strains are expressed in the plate x-y axes (see Figure M7.1.7). Thus the strains in each ply can be found from the transformations of equation, i.e.

$$T^{-1} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & \left(m^2 - n^2\right) \end{bmatrix}$$

or,

$$\bar{\varepsilon}_{12} = T\bar{\varepsilon}xy \qquad (7.1.9)$$

Finally the ply stresses are obtained from the stiffness matrix equation, i.e.

$$\sigma_{12} = Q\varepsilon_{12} \qquad (7.1.10)$$

By applying, on a ply-by-ply basis, a selected failure criterion the occurrence of failure can be determined.

7.1.3.2 Final failure and strength

The final, or ultimate, failure load of an angle ply laminate is often coincident with, or only slightly higher than, the load to cause initial failure. This is not necessarily the case for other lay-ups and final failure can be at a considerably higher load than that to cause first ply failure.

It is clear that once a ply has sustained failure its stiffness in certain directions will have been reduced. However, unless the damaged ply has completely delaminated from the rest of the laminate it will still contribute to the overall stiffness of the plate. The magnitude of this contribution depends on the amount of damage, the fibre/matrix combination and the nature of the loading on the ply.

In general an iterative method is adopted, successively applying the approach described in section 7.1.3.1 until final failure has occurred. As a starting point the relative value of the forces and moments $(N_x, N_y, N_{xy}, M_x, M_y, M_{xy})$ acting on the laminate will be known from, say, a structural analysis of the component being designed. Alternatively a parametric study of a range of lay-ups could be carried out for an arbitrary set of load values. The steps taken are as follows.

1. Apply, in the previously-determined ratio, a small value of the loads to the laminate under consideration.
2. Using laminate analysis determine the plate strains and curvatures and hence the stresses and strains in each ply (in the principal directions).
3. Apply the chosen failure criterion to each ply.
4. If no failure has occurred increase the loads (maintaining their relative magnitudes) by the appropriate factor to give first ply failure.
5. Reduce the stiffnesses in the damaged plies and recalculate the A, B and D matrices.
6. Repeat steps 2 and 3 until no further failures occur.
7. Repeat steps 4, 5 and 6.
8. Repeat step 7 until failure (i.e. fibre fracture) of the last ply takes place. This defines the ultimate strength of the laminate.

It is when applying failure criteria in the above procedure that the choice between limit and interactive methods becomes important. As we have seen, the former tells us the **mode of failure**. Thus, when we come to step 5, we know which stiffnesses to reduce if we use either the maximum stress or maximum strain criterion. If, on the other hand, we were to use the Tsai-Hill criterion we would have to infer the mode of failure from the relative magnitudes of the terms in equation 7.1.4.

Even when we know the failure mode it is still necessary to select the reduction factor for the stiffness terms; at the moment there is no universally accepted approach. Suppose, for example, that step 4 of our procedure indicates transverse tensile failure in a particular ply. One approach would be to put the corresponding stiffness, E_{22} together with ν_{21}, to zero. An alternative would be to reduce the original values by, say, 50% or 90%; some researchers have found the former appropriate to CFRP and the latter to GFRP. The simplest, and least realistic, approach would be to completely disregard the damaged layer. In this case, when recalculating the A, B and D matrices, the laminate would appear to have empty space in place of these layers.

7.1.4 Additional Factors

7.1.4.1 Hygrothermal effects

It was noted in an earlier chapter that some high performance fibres, notably carbon and Kevlar, have small, or even negative, coefficients of thermal expansion. As a consequence, in a unidirectional ply, when cooled to room temperature after curing, the fibres will be in compression and the matrix in tension (parallel to the fibres).

It is easy to see that a complex state of stress will exist in a laminate at room temperature because, in addition to the effect described above, the relatively large transverse contraction of a free ply will be constrained by other plies having differently oriented fibres. These so-called residual thermal stresses can be a significant fraction of the failure stress of the matrix. It is, therefore, vital that such issues are included in any strength prediction procedure.

Similar effects are seen due to the absorption of moisture by a polymer matrix. The resultant swelling is akin to thermal expansion and can be treated in the same way by modification of the laminated plate constitutive equations. Such hygrothermal topics will not be addressed here.

7.1.4.2 Edges

Classical laminate theory (CLT) applies only to plates that are infinitely long and wide. In other words it ignores edges. In many real situations laminates will have edges, for example a plate containing a hole or a plate of finite width.

At such edges the assumptions of CLT break down and the in-plane stresses $(\sigma_x, \sigma_y, \sigma_{xy})$ alone are found not to satisfy local equilibrium on the stress-free boundaries. The through-thickness direct σ_z and shear (τ_{xz}, τ_{yz}) stresses (Figure M7.1.4) can be calculated from advanced applied mechanics theories or from finite element analysis. It is found that within one plate thickness from the edge these stresses can be sufficiently high to exceed the (low) through-thickness strengths. Both their magnitude and sense (tension or compression) are determined by the laminate's stacking sequence. Hence it is possible to minimize their effect and reduce the chance of failure, by interlaminar shear or tension, initiating at an edge.

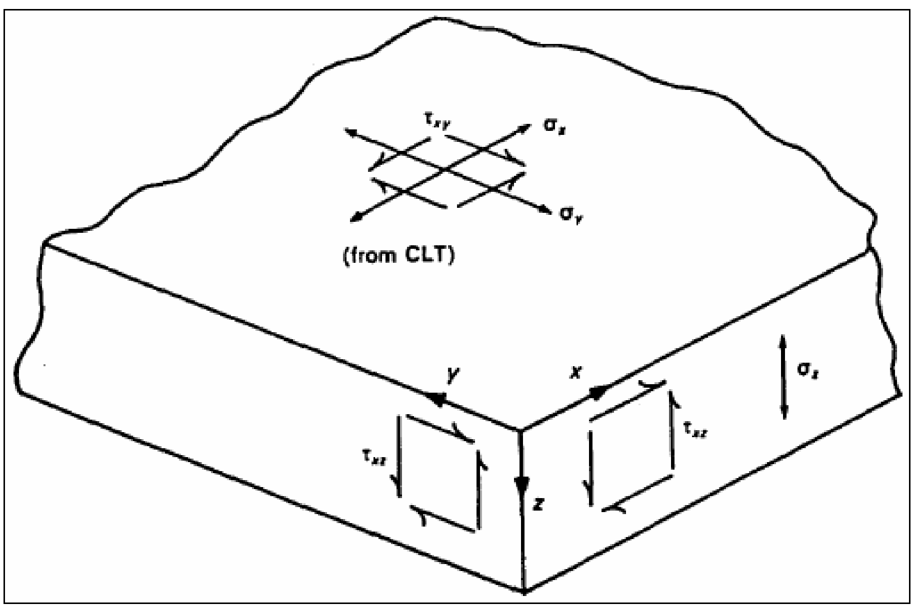


Figure M7.1.4 Stresses at free edge of a laminate: $\sigma_z, \tau_{xz}, \tau_{yz}$ (not predicted by CLT), in the thickness (z) direction.

A simple illustration of this situation is given by a cross-ply laminate when loaded in tension parallel to the 0° plies. With a $(0/90°)_s$ stacking sequence the through-thickness direct stress on the edges parallel to the load is tensile, whilst for a $(0/90°)_s$ stacking sequence the stress is compressive. Clearly the former stacking sequence is to be preferred.

7.1.5 Summary

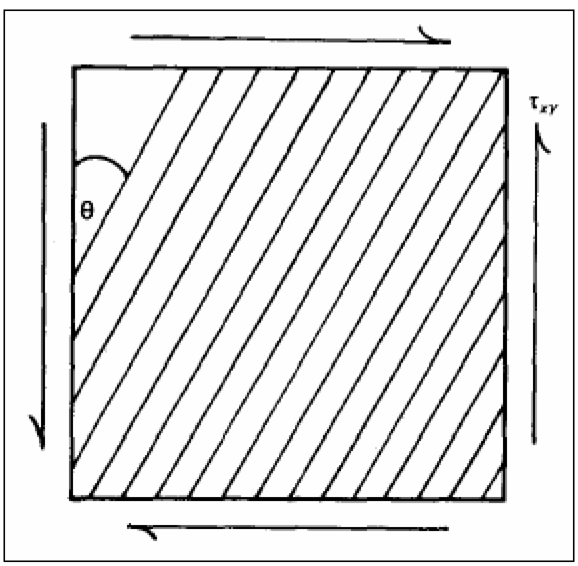
In this chapter we have considered ways of calculating the strength of a lamina or a laminate when subjected to a set of loads (forces and moments) that cause in-plane stresses. The general approach is to obtain from CLT the stresses, or strains, in the principal directions of each ply and to compare these with some limiting values. The comparison is made via a failure criterion.

A large number of failure criteria have been developed, although many of them apply only in restricted circumstances (a given fibre/matrix combination and/or set of loads). Here we looked at the two limit criteria, maximum stress and maximum strain, and one interactive criterion, the Tsai-Hill method.

Initial, or first-ply, failure of a laminate can readily be predicted using a criterion, but the way of predicting final failure is still open to debate. Finally, a full analysis should include hygrothermal effects and, where appropriate, the additional stresses that arise at edges.

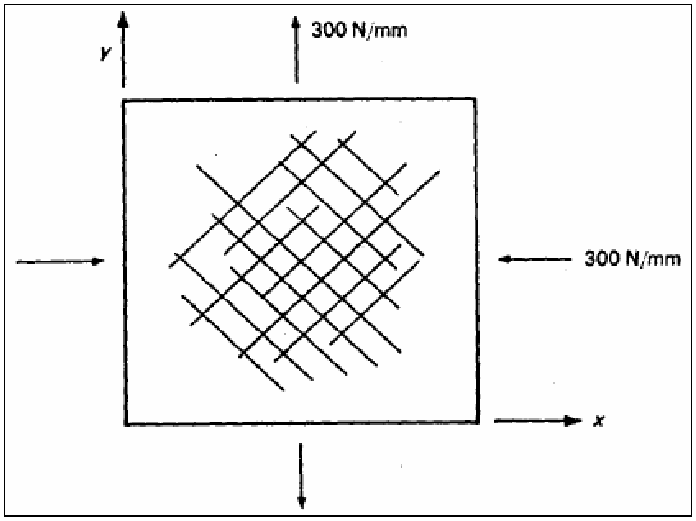
Problems

1. A unidirectional lamina, with the properties given below, is subjected to a shear stress as illustrated in the figure bellow. Determine the magnitude of this stress to cause failure according to the Tsai-Hill criterion when θ=10°, 30° and 60°. E_{11}= 130.0, E_{22} = 10.0, G_{12} = 5.0GPa; ν_{12} =0.3; $\hat{\sigma}_{1T}$ = 1500, $\hat{\sigma}_{2T}$ = 50, $\hat{\sigma}_{1C}$ = 1300, $\hat{\sigma}_{2C}$ = 120, $\hat{\tau}_{12}$ = 80MPa.



2. A 4-ply laminate with $(45^{\circ})_S$ lay-up is subjected to the set of loads shown in the figure bellow. Each ply is 0.125 mm thick with the elastic properties E_{11}=1380GPa, E_{22} =8.96,

G_{112} =7.10 GPa; ν_{12} =0.30 and the strengths of Problem 1(above problem) Determine the reserve factor appropriate to the given loads according to the maximum stress criterion.



<u>Self-Assessment Questions</u>

Indicate whether statements 1 to 6 are true or false.

1. According to the maximum stress criterion lamina failure is determined by the absolute maximum component of stress in the lamina.
   (A) true
   (B) false
2. The maximum stress and maximum strain criteria will predict the same failure loads.
   (A) true
   (B) false
3. A hybrid stress criterion is used for composites containing more than one fibre type (i.e. a hybrid composite).
   (A) true
   (B) false
4. An interactive stress criterion cannot directly predict the mode of failure.
   (A) true
   (B) false
5. An interactive criterion will always predict failure stresses different to those predicted by the maximum stress criterion.
   (A) true
   (B) false
6. When predicting the final failure of a laminate it is necessary to know the failure mode of individual plies.
   (A) true
   (B) false

For each of the statements of questions 7 and 8, one or more of the completions given are correct. Mark the correct completions.

7. Maximum strain criterion
   (A) can be obtained from the maximum stress criterion by dividing each term by an appropriate stiffness,
   (B) comprises five sub-criteria,
   (C) cannot predict failure stresses,
   (D) will give the same prediction as the maximum stress criterion for a lamina in a uniaxial stress state when the stress is parallel to the fibres,
   (E) gives a better prediction than the maximum stress criterion when the stress-strain relation shows significant nonlinearity.
8. Tsai-Hill criterion
   (A) is only applicable if the direct stresses are tensile,
   (B) cannot predict the mode of failure,
   (C) gives better prediction than does a limit criterion for a unidirectional lamina when the fibres are not aligned with the applied stress,
   (D) cannot be used to predict the final failure of a laminate,
   (E) cannot be used to obtain a reserve factor.

Each of the sentences in questions 9 to 15 consists of an assertion followed by a reason.
Answer:

(A) if both assertion and reason are true statements and the reason is a correct explanation of the assertion,
(B) if both assertion and reason are true statements but the reason is not a true explanation of the assertion,
(C) if the assertion is true but the reason is a false statement,
(D) if the assertion is false but the reason is a true statement,
(E) if both the assertion and reason are false statements.

9. A lamina is deemed to have failed when the fibres fracture because the fibres carry the highest stresses.
10. When predicting the failure of an off-axis lamina it is necessary to calculate the stresses in the principal directions because these stresses are always greater than the applied stresses.
11. The maximum stress criterion will always predict failure in tension because the longitudinal tensile strength of a unidirectional ply is greater than the corresponding compressive strength.
12. The Tsai-Hill criterion gives a more accurate prediction for off-axis loading because it does not predict the mode of failure.
13. Initial failure of a cross-ply laminate can only be predicted by the Tsai-Hill criterion because it corresponds to transverse ply cracking.
14. Prediction of laminate failure requires an iterative approach because ply stiffnesses are modified as failures occur.
15. Classical Laminate Theory cannot predict failure of finite width laminates because it ignores the existence of through-thickness stresses.

Answers
Problems
Self-assessment
1- B; 2 - B; 3 - B; 4 - A; 5 - B; 6 - A; 7 - B, D, E; 8 - B, C; 9 - B; 10 - C; 11 - D; 12 - B; 13 - D; 14 - A; 15 - A.

Learning Unit-2: M7.2

M7.2: Failure Mechanics of Composites

Failure in composites shows a much wider variety of mechanisms than that in metals. There are many different modes of failure. There are not a definite number of material's strength properties because it depends on the failure criterion adopted. However, for an orthotropic material, the following 9 strength properties are necessary (not always sufficient though), tensile and compressive strengths in the three principal directions of the material and shear strengths in the three directions. These properties are easily measurable and there are standard procedures for measuring them. Some criteria may require more properties that are more difficult to measure than these and standard procedures are often not found.

As composites are mostly used in forms of laminates, 2D stress states are often sufficient for most applications. The above listed 9 failure properties reduced to tensile or compressive and/or shear stresses for one dimensional and for 2D problems.

The difficulties in establishing and validating failure criteria for composites are associated with the definition of failure in composites, which is not always absolutely clear. For example, in a laminate, one or more laminae may be clearly failing, e.g. cracked, yet the laminate could keep on taking load to much higher level. When the laminate is unloaded from such a damaged state, a full recovery of the shape of the laminate is often observed. In composites, material failure sometimes shows structural behaviour, e.g. compression along fibres. The same strength properties for the same material vary with the size of the specimen, sometimes noticeably. Owing to the multiphased nature of composites, failure in one phase does not necessarily imply the same in others, nor of the composite.

One of the most difficult and challenging subjects to which one is exposed to in the mechanics of advanced composites materials involves finding a suitable failure criterion for these systems. This is compounded by our still limited grasp of understanding and predicting adequately the failure of all classes of monolithic materials to which we seek recourse for guidance in establishing descriptive failure criterion for composites. To a large degree of development of such criteria must be associated with philosophical notions of what the concept of failure is about. For example, in most instances failure is perceived to be separation of structural components or material parts of components. This of course need not be the case since the function of the material and/or component may be the design driver and thus, for example, excessive wear in an axle joint may produce a kinetic motion no longer representative of a key design feature. In addition any micromechanical or sub-structural failure features associated with early on failure initiation such as flaws and voids, surface imperfections, and build-in residual stresses are generally neglected in these design type approaches to failure. These mechanisms would serve as design drives for initiating damage and/or degradation in the composites. Thus, failure identification can best be classified along a spectrum of disciplines and in the particular case of composites further subjugated to different levels of definition of failure dependent upon the level of material characterization. To this end, Figure M7.2.1 appears useful for focusing attention on the different levels of failure characterization and disciplines linkage necessary for identity with each failure types. We focus attention in this section on laminate failure criteria which are based upon lamina and micromechanical failure initiators.

At this point it is worthwhile to identify the important features and mechanisms controlling the microfailure of composite system which produces the damage and/or degradation leading to failure in composite laminates. First, it should be emphasized that the matrix and reinforcing fibers, the primary constituents of a composites, have in general widely different strength characteristics. Also, the interface between the fiber and matrix is known to exhibit a response behavior different than that of the bulk matrix in general. The presence of flaws or defects, introduced into the material system during fabrication, may act as stress raises or failure initiators. Thus, any approach to constructing a micromechanics strength theory should by necessity take into account the influence of such factors. Keeping these thoughts in mind, it appears feasible to characterize failure at the micro level by introducing such local failure modes as:

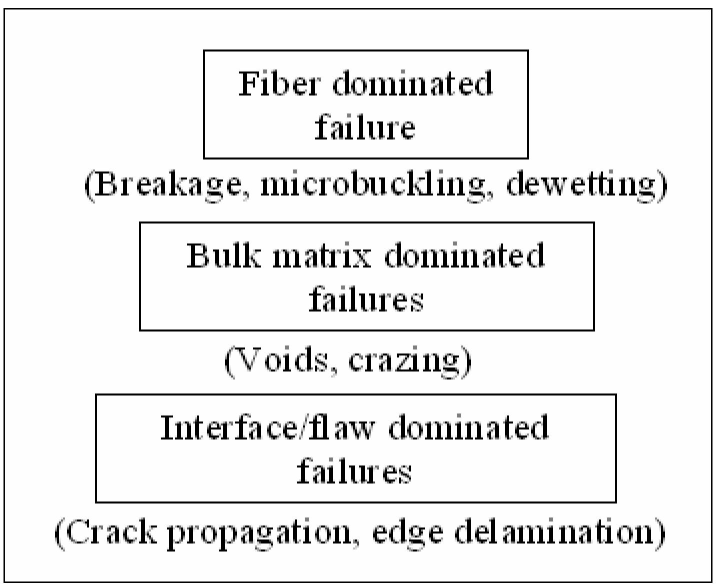


Figure M7.2.1: The different levels of failure characterization and disciplines linkage

Once again, such factors while being important failure initiators are not considered as the primary or causal factors when discussing failure in a global sense. Therefore, as engineers we seek out an observational level of failure to which we can readily relate and feel comfortable with in describing the appropriate failure mechanisms. In this regard we generally attempt to specify lamina failure for anisotropic unidirectional composites or alternatively lamina as existing in composites orthotropic laminates.

To begin this discussion, it has been observed that failure of a laminate considering of a number of plies oriented in different directions relates considering of a number of plies oriented in different directions relative to the loading configuration occurs gradually. This occurs due to the fact that a remaining laminates of the laminate. Although the failure modes can be fiber dominated, matrix dominated or interface/flaw dominated, these basic mechanisms would appear as more important feature to material scientists or material designers. For structural designers, however, the important element to consider is the lamina, made of a chosen fiber-matrix system, which necessary for strength and failure analysis.

The failure theories for composites discussed in learning unit-M7.3 indicate some relevant and distinctive features concerning the strength characteristics of composite materials. A composite lamina is known to exhibit an anisotropic strength behavior, that is, the strength is directionally dependent and thus its tensile and compressive strength are found to be widely different. In addition, the orientation of the shear stresses with respect to the fiber direction in lamina has been observed to have a significant influence on its strength. Finally, another consideration merits attention for laminate failure that the final fracture depends not only on the failure mode or the number of interactive failure modes occurring but also depends on the failure mode which dominates the failure process. These key feature modes which dominates the failure process. These key features of strength and failure of composites makes the subject matter both complex and challenging.

Thus, while we are able to describe the failure of isotropic materials by an allowable stress field associated with the ultimate tensile, compressive and/or shear strength of the material, the corresponding anisotropic (orthotropic) material requires knowledge of at least five principle stresses. These are the longitudinal tensile and the corresponding shear stress. In order to attack failure problem in composites, however, recourse to monolithic material or metals failure criterion are relied upon to serve as role modes for both modification and suitability in predicting failure of laminates. It is therefore worthwhile to review some of the classical failure criteria associated with homogenous isotropic materials to serve as a baseline reference to development of appropriate criteria for the case of failure of homogeneous anisotropic or orthotropic single-ply lamina. Generally speaking, these combined loading failure (strength) theories can be classified into three basic types of criteria as follows:

- Stress Dominated
- Strain Dominated
- Interactive

What to be presented below is some failure criteria most frequently employed in engineering. They are all based on some phenomenological considerations, i.e., macroscopic behaviour instead of the microscopic development of the failure.

Learning Unit-3: M7.3

M7.3 Macromechanical Failure Theories/Failure Theories for Fiber-Reinforced Materials:

M7.3.1 Maximum Stress Criterion

M7.3.1.1 Introduction

Failure of a structural component can be defined as the inability of the component to carry load. Though excessive deformation with the material still intact, as is the case for buckling, can certainly be considered failure in many situations, here failure will be considered to be the loss of integrity of the material itself. In the most basic sense, molecular bonds have been severed. If a fail-safe philosophy has been employed in the design of the structural component, then failure is not necessarily a catastrophic event. Rather, failure causes load redistribution within the structure, a permanent deformation, or some other evidence that load levels have become excessive. The structure is still functional to a limited degree, but steps must be taken if continued use is to be considered.

Failure of fiber-reinforced materials is a complex and important topic, and studies of failure are an ongoing activity. For polymer-matrix composites, because the fiber direction is so strong relative to the other directions, it is clear that failure must be a function of the direction of the applied stress relative to the direction of the fibers. Causing failure of an element of material in the fiber direction requires significantly more stress than causing failure perpendicular to the fibers. Tensile failure in the fiber direction is controlled by fiber strength, while tensile failure perpendicular to the fibers is controlled by the strength of the bond between the fiber and matrix, and by the strength of the matrix itself. But what about the case of a tensile stress oriented at 30° relative to the fibers? We know that for this situation the stress component in the fiber direction, σ_1 the stress component perpendicular to the fibers, σ_2 and the shear stress, τ_{12} can be determined using the stress transformation relations. Which stress component controls failure in this case? Which the stress component in the fiber direction? Which the stress component perpendicular to the fibers? What is the shear stress? Or is it a combination of all three? Because we are now in a position to calculate the stresses in the class of composite structures that satisfy the assumptions of classical lamination theory, it is appropriate to turn to the subject of failure and ask these questions.

There are many issues and controversies surrounding the subject of failure of composite materials. The matrix material of polymer matrix composites may be ductile and exhibit substantial yielding when subjected to high stress levels, and this yielding weakens support of the fiber, or degrades the mechanisms that transfer load into the fibers. On the other hand, the matrix material may be brittle and exhibit significant amounts of cracking around and between fibers as the stress level increases. These crackings will strongly influence the manner and efficiency with which load is transferred into the fibers, and strongly influence the performance of the material. In contrast, failure may be due to the fibers breaking or the fibers debonding and separating from the matrix. Subjected to a compressive load in the fiber direction, the fibers may buckle and deform excessively.

Clearly, we must consider many mechanisms when studying failure. In reality, failure is often a combination of several of these mechanisms, or modes. Failure can simply be the final event in a complex and difficult-to-understand process of damage initiation and accumulation within the material. A structure consists of multiple layers of fiber-reinforced materials, and even multiple materials, and there are multiple fiber directions and a range of load levels and load types. Consequently it is easy to understand why failure of fiber-reinforced composite structures is a difficult topic. Even with a single layer of material, the issues can be quite complicated. As a result, there have been many studies of failure. Each serious user of composite materials tends to develop their own philosophy about failure, based on the application, the material system, and their experience with testing and experimentation. Each large-scale commercial user of composite materials spends much time and capital gathering data to develop criteria and establish design stress levels.

While it is important to understand the mechanisms of failure, for many applications it is impossible to detail each step of the failure process. In the interest of utility, a failure criterion should be reducible to a level that can provide a means of judging whether or not a structure is safe from failure by knowing that a particular stress or combinations of stresses, or combination of strains, is less than some predetermined critical value. The failure criterion should be accurate without being overly conservative, it should be understandable by those using it, and it should be substantiated by experiment. A number of criteria have been proposed; some are rather straightforward and some are quite involved. The maximum stress criterion, maximum strain criterion, and failure criteria that account for interaction among the stress components are commonly used. This is because of the physical bases that underlie these criteria, particularly the maximum stress criterion and the maximum strain criterion. In addition, many of the criteria are simple variations of these, and the variations are based on experimental observation or on slightly different physical arguments put forth by the individuals identified with the criterion.

A legitimate question to ask at this point is, why are there a number of criteria? Isn't one sufficient? The answer is that no one criterion can accurately predict failure for all loading conditions and all composite materials. This is true for isotropic materials—some fail by yielding, others fail by brittle fracture. If we view failure criteria as indicators of failure rather than as predictors in an absolute sense, then having a number of criteria available, none of which covers all situations, becomes an acceptable situation.

In this book we will examine the maximum stress and the Tsai-Wu criteria. These two are chosen because they are among those commonly used for polymer-matrix composites. They represent a divergence of philosophies as to whether or not interaction between stress components is important in predicting failure. Also, by examining any particular criterion, we can present the issues that must be addressed when discussing failure of fiber-reinforced material. By incorporating a particular criterion into a stress analysis of a laminate, failure predictions are possible. The fact that several criteria are commonly used introduces the possibility of determining if using different criteria results in contradictory or similar predictions as to the stress levels permitted and the failure modes expected. Also, a detailed discussion of one or two basic criteria will allow you to form your own opinions regarding failure criteria.

As with the study of the stress-strain behavior of fiber-reinforced material, we shall approach the study of failure of a fiber-reinforced material by examining what happens to a small volume element of material when it is subjected to various components of stress. This is in keeping with the fact that stress is defined at a point, so logically we must assume that failure begins when certain conditions prevail at a point. We will continue to consider the fibers and matrix smeared into an equivalent homogeneous material when we are computing stresses, but to gain insight into the mechanisms that cause failure, it is useful to keep the separate constituents in mind. To that end, consider Figure M7.3.1.1; in the fiber direction, as a tensile load is applied, failure is due to fiber tensile fracture. One fiber breaks and the load are transferred through the matrix to the neighboring fibers. These fibers are overloaded, and with the small increase in load, they fail. As the load is increased, more fibers fail and more load is transferred to the unfailed fibers, which take a disproportionate share of the load. The surrounding matrix material certainly cannot sustain the load and so fibers begin to fail in succession; the failure propagates rapidly with increasing load. As with many fracture processes, the tensile strength of graphite fiber, for example, varies from fiber to fiber; and along the length of a fiber. The tensile strength of a fiber is a probabilistic quantity, and the mean value and its variance are important statistics. It is possible to study the failure of fiber-reinforced composites from this viewpoint, or to simply use a value of failure stress in the fiber direction that includes a high percentage of the fiber failure strengths. That will be the approach here. The tensile strength in the fiber direction will be denoted as σ_1^T.

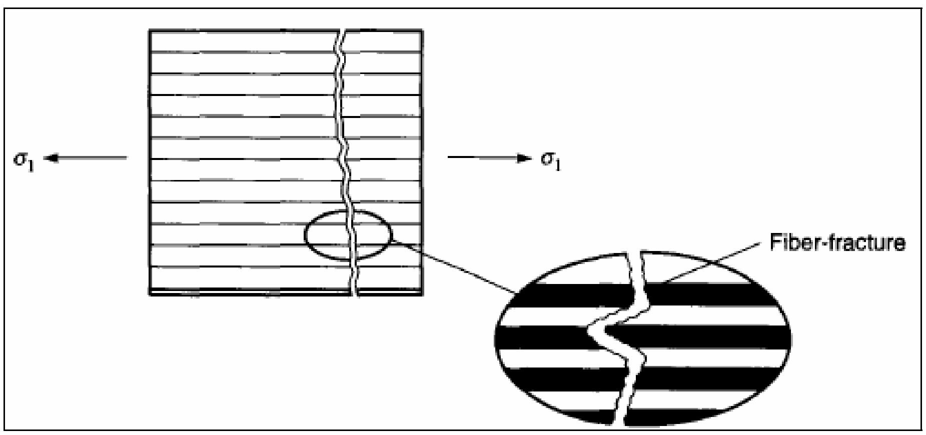


Figure M7.3.1 Failure in tension in the 1 direction

With a compressive stress in the fiber direction, contemporary polymer-matrix composites fail by fiber kinking, or microbuckling, as in Figure M7.3.1.2. Kinking occurs among localized bands, or groups, of fibers, and the fibers in the band fracture at both ends of the kink; the fracture inclination angle is denoted as β, which varies from 10 to 30° in most composites. The width of the kink band, W, varies from 10 to 15 fiber diameters. The primary mechanism responsible for this behavior is yielding, or softening, of the matrix as the stresses within it increase to suppress fiber buckling. As might be suspected, any initial fiber waviness or misalignment, denoted as $\bar{\phi}$ in Figure M7.3.1.2, greatly enhances kinking and reduces fiber-direction compressive strength σ_1^C. In well-made composites, $\bar{\phi}$ is typically between 1 and 4° (0.017-0.070 rads). The study of kinking is an ongoing topic of research, where such issues as the magnitude of the fiber modulus relative to the magnitude of the matrix modulus, bending effects in the fiber, the variance of the misalignment angle, and the influence of other stresses (say, a compressive σ_2), are being studied. It is generally accepted, however, that fiber misalignment and yielding of the matrix influence composite compressive strength in the fiber direction, σ_1^C by way of the relation

$$\sigma_1^C = \frac{G_{12}}{1 + \dfrac{\bar{\phi}}{\gamma_{12}^Y}} \qquad (7.3.1.1)$$

where γ_{12}^Y the shear strain at which the composite shear stress-shear strain relation loses validity due to softening, or yielding, effects in the matrix. The $\bar{\phi}/\gamma_{12}^Y$ values range from 2 to 6, depending on the material.

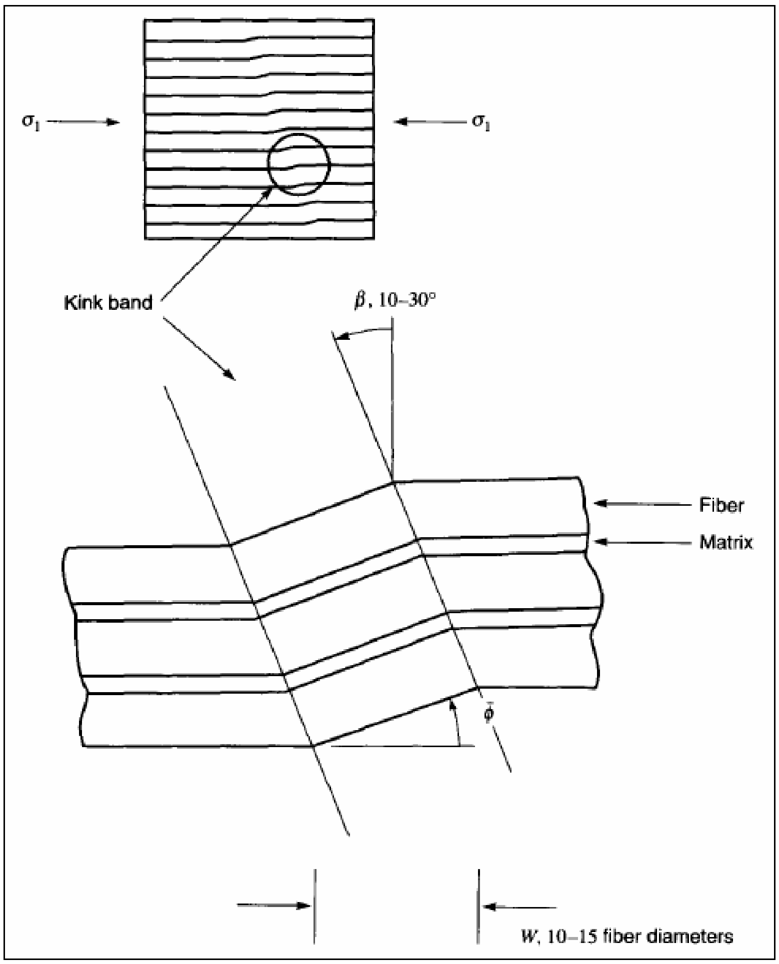


Figure M7.3.1.2 Failure in compression in the 1 direction

Often polymeric fibers in a polymeric matrix fail in compression due to fiber crushing rather than fiber kinking. The compressive stresses in the fiber cause the fiber to fail before the matrix softens enough to allow kinking.

As illustrated in Figure M7.3.1.3, perpendicular to the fiber, say, in the 2 direction, failure could be due to a variety of mechanisms, depending on the exact matrix material and the exact fiber. Generally a tensile failure perpendicular to the fiber is due to a combination of three possible micromechanical failures: tensile failure of the matrix material; tensile failure of the fiber across its diameter, and failure of the interface between the fiber and matrix. The latter failure is more serious and indicates that the fiber and matrix are not well bonded. However, due to the chemistry of bonding, it is not always possible to have complete control of this bond.

Failure in compression perpendicular to the fibers, as in Figure M7.3.1.4, is generally due to material crushing, the fibers and matrix crushing and interacting. The compressive failure stress perpendicular to the fibers is higher than the tensile failure stress in that direction. Herein the tensile failure stress perpendicular to the fibers will be denoted as σ_2^T, while the compressive failure stress will be denoted as σ_2^C. Failure in the 3 direction is similar to failure in the 2 direction and the failure stresses will be denoted as σ_3^T and σ_3^C.

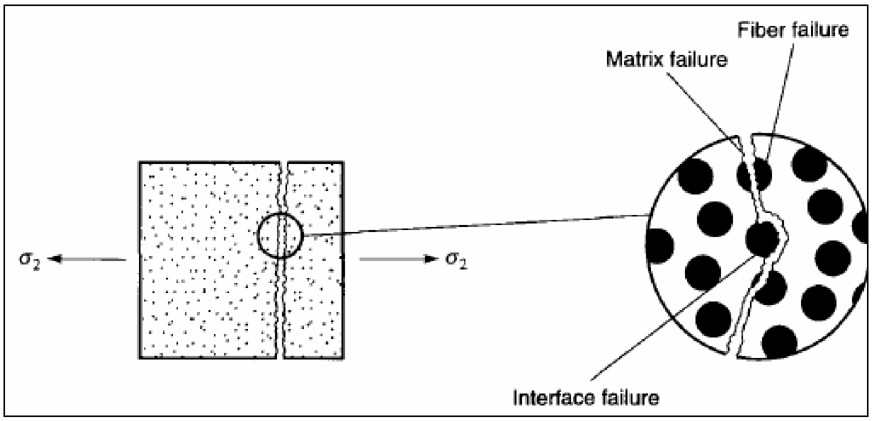


Figure M7.3.1.3 Failure in tension in the 2 direction

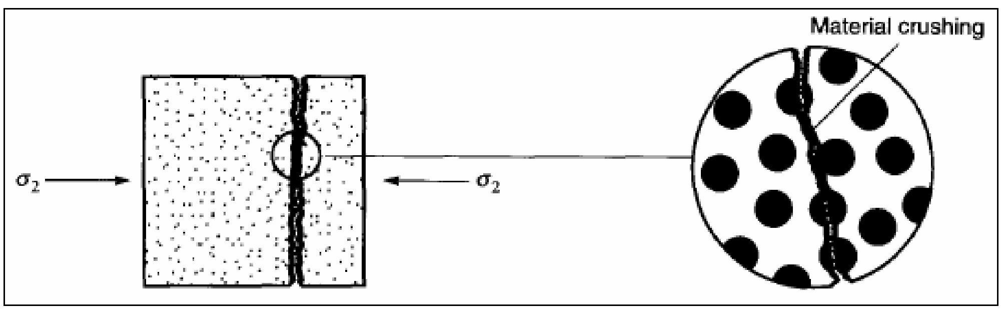


Figure M7.3.1.4 Failure in compression in the 2 direction

The shear strength in the 2-3 planes, denoted as τ_{23}^F, is limited by the same mechanisms that govern tensile strengths perpendicular to the fibers, namely, matrix tensile failure, and failure across the diameter of the fiber, and interfacial strength. Because a shear stress produces a tensile stress on a plane oriented at 45°, these tensile micromechanisms again limit the performance of the material. Figure M7.3.1.5 illustrates these mechanisms as viewed in this shear mode. Because of these mechanisms that control shear strength, the shear strength in the 2-3 plane is independent of the sign of the shear stress.

The shear strength in the 1-2 planes is limited by the shear strength of the matrix, the shear strength of the fiber, and the interfacial shear strength between the fiber and matrix. Figure M7.3.1.6 depicts failure in shear in the 1-2 planes due to a shear separation of the fiber from the matrix along the length of the interface. Failure in the 1-3 plane follows similar reasoning, and these shear strengths are denoted as τ_{12}^{F} and τ_{13}^{F}. As expected, the failure strength in this plane is independent of the sign of the shear stress.

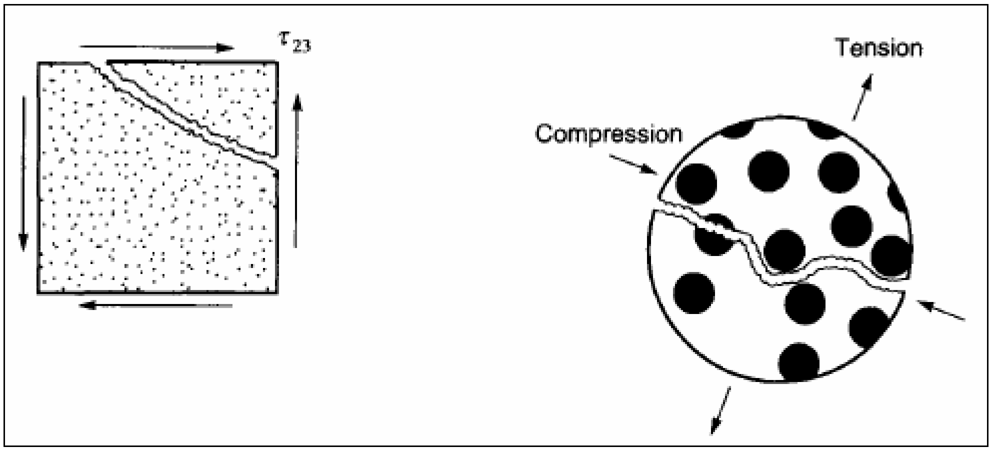


Figure M7.3.1.5 Failure in shear in the 2-3 planes

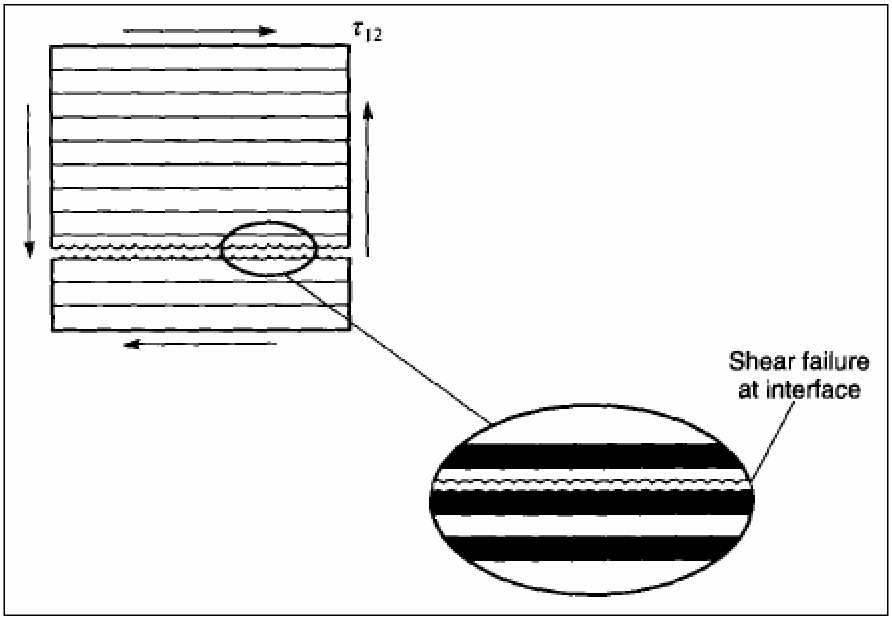


Figure M7.3.1.6 Failure in shear in the 1-2 planes

In summary, then, for a fiber-reinforced composite material there are nine fundamental failure stresses to be concerned with, six normal stresses, three tensile and three compression, and three shear stresses. Each failure stress represents distinct microfailure mechanisms, and each result in a failure that is somewhat unique to that loading situation. Measuring these failure stresses is difficult. Many issues are involved with the testing of fiber-reinforced composites to determine failure stresses; most of them focus on the fixturing and specimen shape and dimensions. There is also the issue of interaction among stress components. We have indicated that compression failure in the fiber direction is due to kinking and microbuckling, and it stands to reason that a compressive stress perpendicular to the fibers—that is, a σ_2 or σ_3—might help support the fibers and prevent, or at least delay, microbuckling and increase the compressive load capacity in the fiber direction. However, testing to study interaction between stress components is difficult. To determine if a compressive stress perpendicular to the fibers increases the compressive strength in the fiber direction, a fixture to vary the level of compressive σ_2 and then the level of compressive σ_1 would have to be constructed and a range of load levels used. The construction of such a biaxial compressive test fixture is difficult. And what is about of triaxial compression? If a compressive stress σ_2 might increase the compressive capacity in the fiber direction, what is about a combination of compressive σ_3 and a compressive σ_2? And what is about the possible interaction of tension perpendicular to the fibers and shear? From Figures 7.3.1.3 and 7.3.1.5 we saw that the same mechanisms that are responsible for limiting the value of τ_{23} limit the value of a tensile σ_2, SO it is conceivable that these two stress components interact to influence failure. The possibilities are enormous. Unfortunately, it is quite easy to generate failure criteria that are too complex to be verified experimentally. An important requirement of any failure criterion is to be able to conduct failure tests on simple specimens subjected to fundamental stress states, and then be able to predict the load levels required to produce failure in more complicated structures with more complex stress states. The two criteria considered here rely on the fundamental failure strengths discussed above. The maximum stress criterion is a noninteractive failure theory, while the Tsai-Wu criterion is an interactive theory, and the influence of stress component interaction will be observed.

Failure stresses (MPa) for graphite and glass composites

	Graphite-reinforced	Glass-reinforced
σ_1^C	−1250	−600
σ_1^T	1500	1000
σ_2^C	−200	−120
σ_2^T	50	30
τ_{12}^F	100	70

Table M7.3.1.1 Failure stresses (MPa) for graphite and glass composites

Because the majority of what we have discussed so far has been devoted to situations where the plane-stress assumption has been used, we shall limit our discussion of failure to those cases also. Hence we shall be interested in the following five failure stresses:

σ_1^C : compression failure stress in the 1 direction
σ_1^T : tensile failure stress in the 1 direction
σ_2^C : compressive failure stresses in the 2 direction
σ_2^T : tensile failure stress in the 2 direction
τ_{12}^F : shear failure stress in the 12 plane (7.3.1.2)

Table M7.3.1.1 gives typical values of these stresses for a graphite-fiber composite and a glass-fiber composite. It stands to reason that the failure levels associated with σ_3 and τ_{13} can be equated to the levels for σ_2 and τ_{12}, respectively, and that the failure stress τ_{23}^F is similar to the other shear failure stresses.

M7.3.2 Definition of 'Maximum Stress failure Criterion'

The maximum stress failure criterion, as it applies to the plane-stress case, can be stated as:
"A fiber-reinforced composite material in a general state of stress will fail when:

Either,

The maximum stress in the fiber direction equals the maximum stress in a uniaxial specimen of the same material loaded in the fiber direction when it fails;

OR,

The maximum stress perpendicular to the fiber direction equals the maximum stress in a uniaxial specimen of the same material loaded perpendicular to the fiber direction when it fails;

OR,

The maximum shear stress in the 1-2 planes equals the maximum shear stress in a specimen of the same material loaded in shear in the 1-2 planes when it fails.

Note the either-or nature of the criterion. Failure can occur for more than one reason. In addition, the first two portions of the criterion each involve tension and compression. Symbolically, the maximum stress criterion states that a fiber-reinforced material will not fail if at every point

$$\sigma_1^C < \sigma_1 < \sigma_1^T$$
$$\sigma_2^C < \sigma_2 < \sigma_2^T$$
$$|\tau_{12}| < \tau_{12}^S \qquad (7.3.1.3)$$

While satisfaction of the inequalities of equation (7.3.1.3) guarantees, according to the criterion, no failure, it is the equalities associated with equation (7.3.1.3) that are important for determining failure loads. These equalities are,

$$\begin{aligned}
\sigma_1 &= \sigma_1^C \\
\sigma_1 &= \sigma_1^T \\
\sigma_2 &= \sigma_2^C \\
\sigma_2 &= \sigma_2^T \\
\tau_{12} &= -\tau_{12}^F \\
\tau_{12} &= \tau_{12}^F
\end{aligned} \qquad (7.3.1.4)$$

Equation (7.3.1.4) defines the boundaries of the no-failure region in principal material coordinate system stress space $\sigma_1 - \sigma_2 - \tau_{12}$. In this space each of the above equations defines a plane, and the totality of the planes defines a rectangular volume. Because of the differences in the tensile and compression failure loads, the geometric center of the volume does not coincide with the origin of the stress space. Figure M7.3.1.7 illustrates this rectangular volume, and it is important to note the proportions in the figure: the rectangular region is much longer in the 1 direction than in the other two directions. In fact, to scale, the rectangular box is much longer and narrower than is depicted in Figure M7.3.1.7. Even though we are dealing with a plane-stress problem, we must resort to a three-dimensional figure to describe the problem. Obviously, illustrating the case for a problem that involves all six components of stress becomes a challenge.

To illustrate the application of the maximum stress criterion and to establish a procedure for using it in such a way that all possible failures in all layers are accounted for, three example failure problems will be studied in detail. These examples will be based on a fiber-reinforced tube, or cylinder, loaded in tension and in torsion. In the first example, the cylinder will be loaded with an axial load, and the question will be to determine the level of load required to produce failure. The second example is a combined load problem: A fixed amount of axial load is applied, and the question is to determine the level of torsional load that can be applied before failure occurs. The third example is also a combined-load problem: The question is to determine what combined levels of torsional and axial load can be applied before failure occurs. This third example is a realistic design case; we determine the torsion-axial load envelope. This third example is quite involved but illustrates the complexity of a failure analysis for a fiber-reinforced composite material.

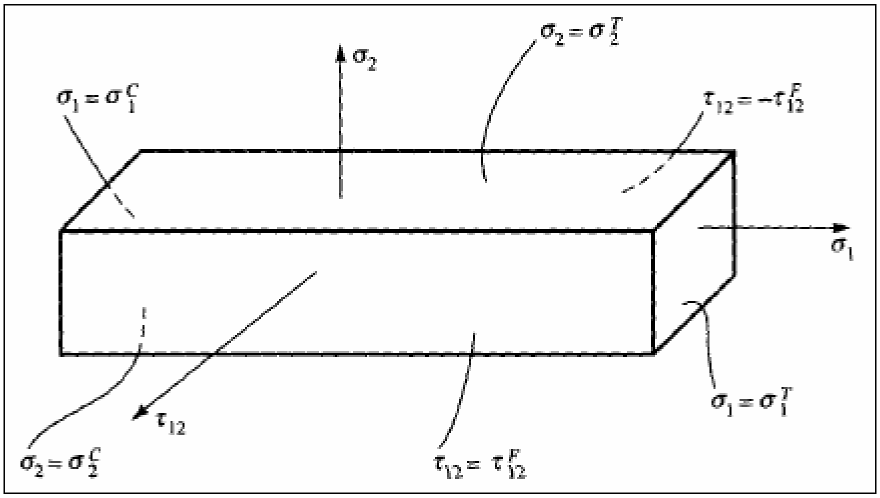


Figure M7.3.1.7 Maximum stress criterion in principal material system stress space

M7.3.1.2 Failure Example 1: Tube with Axial Load —Maximum Stress Criterion

Consider a tube with a mean radius of 25 mm made of graphite-reinforced composite with a 10-layer wall with a stacking sequence of $\left[\pm 20/03\right]_S$. The tube is designed to resist an axial load but has the low-angle off-axis layers (the ±20° layers) to provide some circumferential and torsional stiffness, and to hold the load-carrying layers together. If we use the maximum stress failure criterion, what is the maximum allowable axial load? What layer or layers control failure? What is the mode of failure? This tube is illustrated in Figure M7.3.1.8, and the applied axial load is being denoted as P. The 25-mm mean radius is illustrated, as is the 1.50-mm wall thickness that results from using 10 layers, each of 0.150 mm thickness. We will assume that conditions within the tube wall do not vary with distance along the tube. The procedure recommended throughout our analysis of failure will be to first determine the principal material systems stresses in each layer that are caused by a unit applied load, in this first example a load of P = 1 N. It will then assume that the unit applied load is multiplied by a scale factor. Because we are dealing with a linear problem, the three components of stress in each layer will be multiplied by this same scale factor. These scaled stresses will then be used in the first equation of equation (7.3.1.4) to determine the value of the scale factor that causes failure in compression in the fiber direction in the first layer in the tube. The second equation of equation (7.3.1.4) will then be used to determine the value of the scale factor that causes tensile failure in the fiber direction in the first layer. Then the two equations representing failure perpendicular to the fibers will be checked, then the two shear equations. There will be six values of the scale factor for the first layer in the tube. We then repeat the analysis for the second layer, and six more scale factors result. For the third and subsequent layers, we will compute more scale factors. With the six values of the scale factors for all the layers in hand, the numerical value of the load to cause failure in the laminate, in this case the tube wall is given by the value of the smallest scale factor. By keeping track of the scale factors, it is possible to determine which layer or layers are responsible for failure, and what limits the load capacity (fiber tension, tension perpendicular to the fibers, etc.). This approach, though tedious, can be programmed into a computer-based laminate analysis and the scale factors computed, sorted, listed in ascending order, and correlated to the specific layers and specific failure modes.

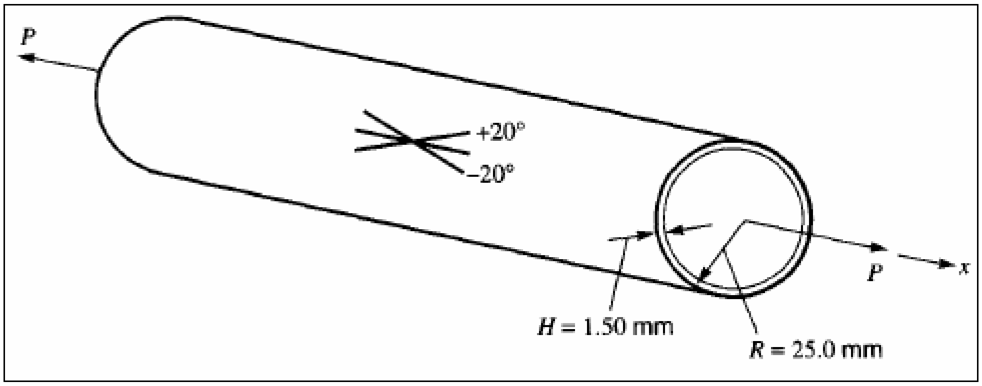


Figure M7.3.1.8 Tube with axial load P, Failure Example 1

Though a tube should be analyzed in a polar coordinate system, let us use our rectangular notation and assign x to the axial direction, v to the circumferential direction, and z to the outward radial direction. To study the example problem, let us not be overly concerned with the method of transmitting the load P into the tube. We shall assume that P is distributed uniformly around the circumference of the ends of the tube, and thus, in keeping with the definition of N_x as being a load per unit length of laminate, N_x is the load P divided by the circumference of the end; that is:

$$N_x = \frac{P}{2\pi R} \qquad (7.3.1.5)$$

This is the only load acting on the laminate constituting the tube wall and thus

$$\begin{aligned} N_x &= \frac{P}{2\pi R} \\ N_y &= 0 \\ N_{xy} &= 0 \\ M_x &= 0 \\ M_y &= 0 \\ M_{xy} &= 0 \end{aligned} \qquad (7.3.1.6)$$

With this approach it is being assumed that the tube acts like a rolled-up flat laminate, and every point on the reference surface, in this case the mean radius of the tube, is subjected to the force resultants of equation (7.3.1.6). This approach leads to quite accurate answers as long as the ratio of the radius to wall thickness is greater than 10, and the details of the load introduction at the ends of the tube are not important. If we assume that P = 1 N, then, with R- 25 mm, equation (7.3.1.5) leads to

$$N_x = 6.37 \text{ N/m} \qquad (7.3.1.7)$$

Because the laminate is symmetric and balanced, with only force resultants applied, to compute the stresses in the various layers due to this the unit value of P we need only compute the elements of the A matrix. We can invert the A matrix and compute the reference surface strains, and knowing the reference surface strains, we can compute the strains throughout the wall thickness, and then the stresses. In particular, we can compute the stresses in the principal material system for each layer due to this load of P - 1 N.

Table M7.3.1.2.2 presents the stresses in the principal material system for a graphite-fiber reinforced $[\pm 20/0_3]_S$ laminate using the material properties of Table M7.3.1.2.1 and subjected to the stress resultant of equation (7.3.1.7). By the nature of the problem the stresses within each layer are independent of position within that layer.

	Graphite-polymer composite[1]	Glass-polymer composite	Aluminum
E_1	155.0 GPa	50.0 GPa	72.4 GPa
E_2	12.10 GPa	15.20 GPa	72.4 GPa
E_3	12.10 GPa	15.20 GPa	72.4 GPa
ν_{23}	0.458	0.428	0.300
ν_{13}	0.248	0.254	0.300
ν_{12}	0.248	0.254	0.300
G_{23}	3.20 GPa	3.28 GPa	—[2]
G_{13}	4.40 GPa	4.70 GPa	—[2]
G_{12}	4.40 GPa	4.70 GPa	—[2]
α_1	-0.01800×10^{-6}/°C	6.34×10^{-6}/°C	22.5×10^{-6}/°C
α_2	24.3×10^{-6}/°C	23.3×10^{-6}/°C	22.5×10^{-6}/°C
α_3	24.3×10^{-6}/°C	23.3×10^{-6}/°C	22.5×10^{-6}/°C
β_1	146.0×10^{-6}/%M	434×10^{-6}/%M	0
β_2	4770×10^{-6}/%M	6320×10^{-6}/%M	0
β_3	4770×10^{-6}/%M	6320×10^{-6}/%M	0

[1] In the chapters to follow it will be assumed that a layer thickness is 150×10^{-6} m, or 0.150 mm.
[2] $G = E/2(1 + \nu)$.

Table7.3.1.2.1 Typical engineering of several materials

If the axial load applied to the tube is p, then the stresses in each layer in the principal material system are as given in Table M7.3.1.3; these stresses are simply the stresses in Table M7.3.1.2 multiplied by p.

Layer	σ_1	σ_2	τ_{12}
+20°	+3830	−112.3	−148.7
−20°	+3830	−112.3	+148.7
0°	+4770	−168.6	0

Table M7.3.1.2.2 Principal material system stresses (Pa) in axially loaded tube for P = 1 N

Layer	σ_1	σ_2	τ_{12}
+20°	+3830p	−112.3p	−148.7p
−20°	+3830p	−112.3p	+148.7p
0°	+4770p	−168.6p	0

Table M7.3.1.3 Principal material system stresses (Pa) in axially loaded tube for P =p (N)

The values of stresses in terms of p can be used in the maximum stress criterion to determine the value of p that will cause failure. In particular, in the +20° layers, referring to the first equation of equation (7.3.1.3), we see that failure will not occur in the fiber direction if

$$\sigma_1^C < \sigma_1 < \sigma_1^T \qquad (7.3.1.8)$$

or, substituting numerical values, if

$$-1250 \times 10^6 < 3830p < 1500 \times 10^6 \qquad (7.3.1.9)$$

Using these results as in the first equation of equation (7.3.1.4), compression failure in the fiber direction is given by the condition

$$\sigma_1 = \sigma_1^C \qquad (7.3.1.10)$$

or,

$$3830p = -1250 \text{ x } 10^6 \qquad (7.3.1.11),$$

This result in,

$$p = -32\,7\,000 \qquad (7.3.1.12)$$

For failure of the fibers in tension,

$$\sigma_1 = \sigma_1^T \qquad (7.3.1.13)$$

or,

$$3830p = 1500 \text{ x } 106 \qquad (7.3.1.14)$$

This leads to,

$$p = 392\ 000 \quad (7.3.1.15)$$

Thus the +20° layers fail in compression in the fiber direction when the applied load is P = —0.327 MN, and the layers fail in tension in the fiber direction when the applied load is P = +0.392 MN.

We now examine failure in the +20° layers perpendicular to the fibers, that is, due to σ_2. By the second equation of equation (7.3.1.3) the layers are safe from failure due to σ_2 if

$$\sigma_2^C < \sigma_2 < \sigma_2^T \quad (7.3.1.16)$$

Numerically, the above becomes

$$-20\ 0 \times 106 < -112.3/7 < 50 \times 106 \quad (7.3.1.17)$$

Using the equalities from this equation to determine the values of p that cause failure, we find that failure of the layers due to compression in the 2 direction is given by the condition

$$\sigma_2 = \sigma_2^C \quad (7.3.1.18)$$

Numerically,

$$-112.3p = -20\ 0 \times 106 \quad (7.3.1.19)$$

Or,

$$p = 1780\ 000 \quad (7.3.1.20)$$

Failure due to a tensile failure of the material in the 2 direction is given by

$$\sigma_2 = \sigma_2^T \quad (7.3.1.21)$$

$$-122.3p = 50 \times 106 \quad (7.3.1.22)$$

or,

which leads to,

$$p = -44\ 5\ 000 \quad (7.3.1.23)$$

From these results, we can conclude that a tensile load of P = +1.780 MN on the tube causes the +20° layers to fail in compression in the 2 direction, while a compressive load of P = -0.445 MN on the tube causes the +20° layers to fail in tension in the 2 direction. It is important to keep track of the signs so that proper interpretation of the results is possible.

A shear failure in the +20° layers is given by the condition from equation (7.3.1.3) of

$$-\tau_{12}^F < \tau_{12} < \tau_{12}^F \qquad (7.3.1.24)$$

or, with numerical values,

-10 0 x 106 < -148.7p < 100 x 106 (7.3.1.25)

From this it can be concluded that

p = 673 000 (7.3.1.26)

will cause the +20° layers to fail due to $-\tau_{12}$ and

p = -67 3 000 (7.3.1.27)

will cause the +20° layers to fail due to $+\tau_{12}$.

Table M7.3.1.4 summarizes the results from the analysis of the +20° layers. From the table we can deduce that failure in the +20° layers can be caused by a compressive load P of -0.327 MN due to a compressive stress in the fiber direction, or by a tensile load P of +0.392 MN, due to a tensile stress in the fiber direction.

Having determined the failure characteristics of the +20° layers, we have completed one-third of the failure analysis. Failure analysis of the -20° and 0° layers follows similar steps. For the -20° layers the steps are as follows.
Compression failure in the 1 direction:

$$\begin{aligned} 3830p &= -1250 \times 10^6 \\ p &= -327\,000 \end{aligned} \qquad (7.3.1.28)$$

Tension failure in the 1 direction:

$$\begin{aligned} 3830p &= 1500 \times 10^6 \\ p &= 392\,000 \end{aligned} \qquad (7.3.1.29)$$

Failure mode					
σ_1^C	σ_1^T	σ_2^C	σ_2^T	$-\tau_{12}^F$	$+\tau_{12}^F$
–0.327	+0.392	+1.780	–0.445	+0.673	–0.673

Table M7.3.1.4 Loads P (MN) to cause failure in +20° layers: Maximum stress criterion

Compression failure in the 2 direction:

$$-112.3p = -200 \times 10^6$$
$$p = 1\,780\,000 \qquad (7.3.1.30)$$

Tension failure in the 2 direction:

$$-122.3p = 50 \times 10^6$$
$$p = -445\,000 \qquad (7.3.1.31)$$

Shear failure due to $-\tau_{12}$:

$$148.7p = -100 \times 10^6$$
$$p = -673\,000 \qquad (7.3.1.32)$$

Shear failure due to $+\tau_{12}$:

$$148.7p = 100 \times 10^6$$
$$p = 673\,000 \qquad (7.3.1.33)$$

Table M7.3.1.5 summarizes the results for the -20° layers, and by comparing the results of this table with the results of Table M7.3.1.4, we see that the +20° and -20° layers have very similar failure characteristics; the only difference is in the sign of the load required to produce failure due to a $+\tau_{12}$ or $-\tau_{12}$ failure mode. The failure analysis of the 0° layers is as follows: Compression failure in the 1 direction:

$$4770p = -1250 \times 10^6$$
$$p = -262\,000 \qquad (7.3.1.34)$$

Tension failure in the 1 direction:

$$4770p = 1500 \times 10^6$$
$$p = 315\,000 \qquad (7.3.1.35)$$

Compression failure in the 2 direction:

$$-168.6p = -200 \times 10^6$$
$$p = 1\ 186\ 000 \qquad (7.3.1.36)$$

Tension failure in the 2 direction:

$$-168.6p = 50 \times 10^6$$
$$p = -297\ 000 \qquad (7.3.1.37)$$

Failure mode					
σ_1^C	σ_1^T	σ_2^C	σ_2^T	$-\tau_{12}^F$	$+\tau_{12}^F$
–0.327	+0.392	+1.780	–0.445	–0.673	+0.673

Table M7.3.1.5 Loads P (MN) to cause failure in —20° layers: Maximum stress criterion

Shear failure due to $-\tau_{12}$:

$$0p = -100 \times 10^6$$
$$p = -\infty \qquad (7.3.1.38)$$

Shear failure due to $+\tau_{12}$:

$$0p = 100 \times 10^6$$
$$p = +\infty \qquad (7.3.1.39)$$

The proper interpretation of the infinite values of applied load required to produce shear failure in the 0° layers is that shear failure cannot be produced in those layers with an applied axial load.

Table M7.3.1.6 summarizes the failure analysis for all the layers in the [±20/03]s tube, and examination of the table shows that a tensile load of P = +0.315 MN causes the 0° layers to fail due to tensile stresses in the 1 direction. Discounting failure of the tube due to overall buckling, a compressive load of P =-0.262 MN causes the 0° layers to fail due to compressive stresses in the 1 direction, presumably due to fiber kinking and microbuckling. These are taken as the failure loads, and the accompanying failure modes, for the tube. As can be seen from this example, failure analysis based on the maximum stress criterion is quite straightforward and, as mentioned before, amenable to automation by computer programming when stresses are being computed.

This first failure example brings to light an important point regarding failure calculations. Considering the compressive failure to illustrate the point, and considering the load to start from zero and be increased in magnitude, we see from Table M7.3.1.6 that a load of P -0.297 MN would cause the 0° layers to fail due to a tensile stress in the 2 direction, if the layers did not fail due to compression in the 1 direction due to a load P = -0.262 MN. We have mentioned the somewhat probabilistic nature of failure, so if only some of the fibers fail in compression at P = -0.262 MN, then the tube could sustain more compressive axial load than P = -0.262 MN. The value of P = -0.297 MN is only 13 percent larger in magnitude than the value P = -0.262 MN. Thus, it is possible, with only some of the fibers failing at P = -0.262 N, that the load could be increased by 13 percent, at which point failure in tension in the 2 direction would begin, say, due to matrix cracking parallel to the fibers. At this load level, it is highly likely that the tube would lose all load-carrying capacity. The value of P = -0.262 MN, then, could be interpreted as the value of the load at which failure first occurs, so-called first-ply failure. Using this value as the load beyond which the tube is incapable of sustaining any more load, then, would be a conservative estimate of load capacity.

Layer	Failure mode					
	σ_1^C	σ_1^T	σ_2^C	σ_2^T	$-\tau_{12}^F$	$+\tau_{12}^F$
+20°	–0.327	+0.392	+1.780	–0.445	+0.673	–0.673
–20°	–0.327	+0.392	+1.780	–0.445	–0.673	+0.673
0°	–0.262	+0.315	+1.186	–0.297	$-\infty$	$+\infty$

Table M7.3.1.6 Summary of loads P (MN) to cause failure in $[\pm 20/30]_S$ tube: Maximum stress criterion

Exercises for Section 7.3.1.2

1. Suppose the off-axis layers in the tube of Failure Example 1 were at ±30° instead of ±20°.

   (a) What would be the axial load capacity of the tube?
   (b) Would the failure mode and the layers that control failure be the same as when the fibers were at ±20°? To answer this question you essentially must redo the example problem, starting with the stresses due to P = 1N (i.e., beginning at Table M7.3.1.2 and proceeding).

2. A [±45/02] s graphite-reinforced plate is subjected to a biaxial loading such that the stress resultant in the y direction is opposite in sign to and one-half the magnitude of the stress resultant in the x direction. Call the stress resultant in the x direction N; the loading is given by

$$N_x = N$$
$$N_y = -0.5N$$
$$N_{xy} = 0$$

   (a) Using the maximum stress criterion, compute the value of N to cause failure,
   (b) What layer or layers control failure?
   (c) What is the mode of failure? To answer these questions, compute the failure loads for each of the layers by using the procedure just discussed in connection with the axially

loaded tube. Put the results in table form, as in Table M7.3.1.6, and answer the questions.

M7.3.1.3 Failure Example 2: Tube in Torsion—Maximum Stress Criterion

This example of a failure analysis begins to address the question of a combined loading by looking at the case of tension and torsion on this same tube. Consider that the tube is designed to resist axial load but in a particular application the axial load is 0.225 MN tension and there is an unwanted amount of torsion, T. With the 0.225 MN tensile axial loads acting on the tube, what is the maximum amount of torsion the tube can withstand before it fails? What layer or layers control failure and what is the mode of failure?

Figure M7.3.1.9 illustrates this case and, as with the previous example, we will not be overly concerned with the way the loads are transmitted to the ends. We will assume that both the axial and torsion loads are distributed uniformly over the ends.

$$\begin{aligned} N_x &= \frac{P}{2\pi R} \qquad & M_x &= 0 \\ N_y &= 0 \qquad & M_y &= 0 \\ N_{xy} &= \frac{T}{2\pi R^2} \qquad & M_{xy} &= 0 \end{aligned} \qquad (7.3.1.40)$$

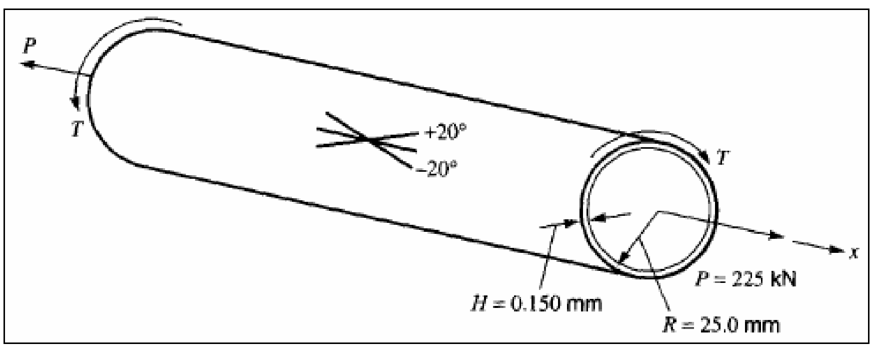


Figure M7.3.1.9 Tube with axial load P = 0.225 MN and torsion T, Failure Example 2

We shall follow the same procedure as before in that we shall examine the stresses in each layer due to unit applied torsion and then use equation (7.3.1.4) to determine the level of torsion required to produce failure in the various modes in the various layers. The added feature in the problem is that the 0.225 MN axial tensile loads is also acting on the tube.

Table M7.3.1.7 gives the stresses in each layer when 0.225 MN of axial load are applied. These are the stresses of Table M7.3.1.2 multiplied by 225 000. Table M7.3.1.8 presents the stresses in each layer when a torsion T = 1 N-m is applied to the tube (no axial load). These are calculated from the condition

$$N_{xy} = 255 \ N/m \qquad (7.3.1.41)$$

Note that in the 0° layers the applied torsion does not produce any stress in the fiber direction or perpendicular to the fiber direction. This is not the case for the ±20° layers. When a torsional load of T = t is applied to the tube, the stresses in each layer are t times the values in Table M7.3.1.8. When both a 0.225 MN tensile load and a torsional load of T = t are applied to the tube, the stresses in each layer are those in Table M7.3.1.9. Superposition of the effects of the axial load alone and the torsional load alone can be used because this problem is strictly a linear problem. The goal of this failure analysis is to find the value or values of t that cause failure of the tube.

As in the previous example, the values of stress, in terms of t, from Table M7.3.1.9 can be used in the maximum stress criterion to determine the value of t that will cause failure. In the +20° layers, referring to the first equation of equation (7.3.1.3), we find that failure will not occur in the fiber direction if

$$\sigma_1^C < \sigma_1 < \sigma_1^F \qquad (7.3.1.42)$$

Layer	σ_1	σ_2	τ_{12}
+20°	+861	−25.3	−33.5
−20°	+861	−25.3	+33.5
0°	+1072	−37.9	0

Table M7.3.1.7 Stresses (MPa) in axially loaded [±20/03]s tube for/' = 0.225 MN

Layer	σ_1	σ_2	τ_{12}
+20°	+0.804	−0.0481	+0.0552
−20°	−0.804	+0.0481	+0.0552
0°	0	0	+0.0721

Table M7.3.1.8 Stresses (MPa) in torsionally loaded [±20/03]s tube for T=\ N m

Layer	σ_1	σ_2	τ_{12}
+20°	+861 + 0.804*t*	−25.3 − 0.0481*t*	−33.5 + 0.0552*t*
−20°	+861 − 0.804*t*	−25.3 + 0.0481*t*	+33.5 + 0.0552*t*
0°	+1072 + 0*t*	−39.7 + 0*t*	0 + 0.0721*t*

Table M7.3.1.9 Stresses (MPa) in [±20/03]s tube loaded with tensile load P = 0.225 MN and torsional load T = t (N m)

or, substituting from Table M7.3.1.9, if

$$-1250 < 861 + 0.0804t < 1500 \qquad (7.3.1.43)$$

Using this as in the first equation of equation (7.3.1.4), we find that compression failure in the fiber direction is given by the condition

$$\sigma_1 = \sigma_1^C \qquad (7.3.1.44)$$

or $\quad 861+0.804t = -1250 \qquad (7.3.1.45)$

This results in,

$$t = -262\ 0 \qquad (7.3.1.46)$$

For failure of the fibers in tension,

$$\sigma_1 = \sigma_1^T \qquad (7.3.1.47)$$

or,

$$861 + 0.804t = 1500 \qquad (7.3.1.48)$$

This leads to,

$$t = 795 \qquad (7.3.1.49)$$

From these results we can say that the +20° layers fail in compression in the fiber direction when, in addition to the applied axial load of P = +0.225 MN, the applied torque is T = -2620 Nm, and the layers fail in tension in the fiber direction when the additional applied torque is T = +795 Nm.

Turning to failure in the +20° layers due to stresses perpendicular to the fibers, we find by the second equation of equation (7.3.1.3) that the layers are safe from failure due to σ_2 if,

$$\sigma_2^C < \sigma_2 < \sigma_2^T \qquad (7.3.1.50)$$

Numerically, the above becomes

$$-20\ 0 < -25.\ 3 - 0.048t < 50 \qquad (7.3.1.51) -$$

Using the equalities from this equation to determine the values of f that cause failure, we find that failure of the +20° layers due to compression in the 2 direction is given by the condition,

$$\sigma_2 = \sigma_2^C \qquad (7.3.1.52)$$

or $\quad t = 3630 \qquad (7.3.1.54)$

Failure due to a tensile failure of the material in the 2 direction is given by,

$$\sigma_2 = \sigma_2^T \qquad (7.3.1.55)$$

or

$$-25.\ 3 - 0.048t = 50 \qquad (7.3.1.56)$$

which leads to,

$$t = -1563 \qquad (7.3.1.57)$$

From these results, we can see that a torsional load of T = +3630 Nm causes the +20° layers to fail in compression in the 2 direction, while a torsional load of T = —1563 Nm causes the +20° layers to fail in tension in the 2 direction.

A shear failure in the +20° layers is given by the condition from equation (7.3.1.3) of

$$-\tau_{12}^{F} < \tau_{12} < \tau_{12}^{F} \qquad (7.3.1.58)$$

or, if we use numerical values,

$$-10\ 0 < -33.5 + 0.0552t < 100 \qquad (7.3.1.59)$$

From this, we can conclude that,

$$t = -1205 \qquad (7.3.1.60)$$

will cause the +20° layers to fail due to $-\tau_{12}$ and that,

$$t = 2420 \qquad (7.3.1.61)$$

will cause the +20° layers to fail due to $+\tau_{12}$. Unlike the case of a tensile load alone, Failure Example 1, the values of torsional load that cause shear failures due to $+\tau_{12}$ and $-\tau_{12}$ differ by more than a sign. The biasing effect of the +0.225 MN axial force applied to the tube causes this result.

The results from the analysis of the +20° layers are summarized in Table M7.3.1.10, and we can see that in the presence of an axial force of +0.225 MN, failure in the +20° layers can be caused by an applied torque T of —1205 N-m. Such a torque will cause the layers to fail due to a negative shear stress τ_{12}. Alternatively, an applied torque T of +795 Nm causes the layers to fail due to tensile stresses in the 1 direction.

Failure mode					
σ_1^C	σ_1^T	σ_2^C	σ_2^T	$-\tau_{12}^F$	$+\tau_{12}^F$
-2620	+795	+3630	−1563	−1205	+2420

Table M7.3.1.10 Torsions T (Nm) to cause failure in + 20° layers with P — + 0.225 MN: Maximum stress criterion

Failure of the —20° and 0° layers follows similar steps. For the —20° layers the steps are as follows. Compression failure in the 1 direction:

$$861 - 0.804t = -1250 \quad (7.3.1.62)$$
$$t = 2620$$

Tension failure in the 1 direction:

$$861 - 0.804t = 1500 \quad (7.3.1.63)$$
$$t = -795$$

Compression failure in the 2 direction:

$$-25.3 + 0.048t = -200 \quad (7.3.1.64)$$
$$t = -3630$$

Tension failure in the 2 direction:

$$-25.3 + 0.0481t = 50 \quad (7.3.1.65)$$
$$t = 1563$$

Shear failure due to $-\tau_{12}$

$$33.5 + 0.0552t = -100 \quad (7.3.1.66)$$
$$t = -2420$$

Shear failure due to $+\tau_{12}$:

$$33.5 + 0.0552t = 100 \quad (7.3.1.67)$$
$$t = 1205$$

The failure analysis of the 0° layers is as follows:

Compression failure in the 1 direction:

$$1072 + 0t = -1250 \quad (7.3.1.68)$$
$$t = -\infty$$

Tension failure in the 1 direction:

1072 + 0t = 1500 (7.3.1.69)

t = +∞

From these last two statements it is clear that failure in the fiber direction in the 0° layers will not occur due to an applied torque. Continuing with the steps in the failure analysis, we find that the remaining steps for the 0° layers are as follows.

Compression failure in the 2 direction:

-39.7 + 0t = -200 (7.3.1.70)

t = -∞

Layer	Failure mode					
	σ_1^C	σ_1^T	σ_2^C	σ_2^T	$-\tau_{12}^F$	$+\tau_{12}^F$
+20°	−2620	+795	+3630	−1563	−1205	+2420
−20°	+2620	−795	−3630	+1563	−2420	+1205
0°	−∞	+∞	−∞	+∞	−1387	+1387

Table M7.3.1.11 Summary of torsions T (Nm) to cause failure in [±20/03]s tube with P = 0.225 MN: Maximum stress criterion

Tension failure in the 2 direction:

-39. 7 +0t = 50 (7.3.1.71)

t = +∞

Like the equations for failure in the 0° direction, these last two statements lead to the conclusion that the 0° layers will not fail in the 2 direction due to the applied torque. Continuing, we find that shear failure due to $-\tau_{12}$ is determined with

0.0721t = -10 0 (7.3.1.72)

t = -1387

and shear failure due to $+\tau_{12}$ with,

0.0721t = 100 (7.3.1.73)

t = 1387

Table M7.3.1.11 summarizes the failure analysis for all the layers, and here we see that a torque of T = +795 Nm causes the +20° layers to fail due to positive a\ and a torque of T = —795 Nm causes the —20° layers to fail also due to a positive σ_1. This torques is taken as the failure torques for the tube. Apparently the effects of the applied torsion added to the +0.225 MN axial load are enough to break the fibers in tension in the +20° layers or the —20° layers, depending on the direction of the applied torsion. Essentially, referring to Figure M7.3.1.7, we find that the initial axial load shifts the origin of the stress space relative to the failure surface.

Exercise for Section 7.3.1.3

1. A [±45/02]s graphite-reinforced plate is subjected to a biaxial loading such that the stress resultant in the y direction is —0.200 MN/m and the stress resultant in the x direction is variable, that is:

$$N_x = N$$
$$N_y = -0.200 \text{ MN/m}$$
$$N_{xy} = 0$$

(a) Using the maximum stress criterion, compute the value of N required to cause failure.
(b) What layer or layers control failure?
(c) What is the mode of failure?

7.3.1.4 Failure Example 3: Tube with Combined Load— Maximum Stress Criterion

As a final example of the application of the maximum stress failure criterion, consider the following problem with the tube: In a particular application the tube is being used with an axial load P and a small torsional load T. According to the maximum stress criterion, what are the ranges of applied axial load and applied torsion the tube can withstand before it fails? What layer or layers control failure and what is the mode of failure? This is truly a combined-load problem, with unknown levels of both the axial and torsional loads, the goal being to find the bounds of P and T within which the tube is safe from failure. The approach here will be to revert to a P-T space and study the failure boundaries. Here P will be on the horizontal axis and T on the vertical axis. For multilayer laminates such as the tube, each layer will have an envelope in this space and the totality of envelopes represents the envelope for the laminate. Figure M7.3.1.10 illustrates the situation being studied.

We shall proceed exactly as before, using a scale factor to multiply the loads and then enforcing the equations representing various portions of the failure criterion to determine the scale factor. Here, however, both the scale factor for the axial load p and the scale factor for the torsional load t are unknown. We can, however, use the six equations of the failure criterion to determine relations between p and t. The stress components in each layer due to a unit axial load, P = 1 N, were given in Table M7.3.1.2, and the stress components in each layer due to a unit torque, T — 1 Nm, were given in Table M7.3.1.8. Accordingly, the stress components in each layer due to the combined effects of an axial load of p and a torsional load of t are given in Table M7.3.1.12. These stress components will now be used in the failure equations, equation (7.3.1.3), and the alternate form, equation (7.3.1.4).

For the +20° layers, the first equation of equation (7.3.1.3), namely,

$$\sigma_1^C < \sigma_1 < \sigma_1^T \qquad (7.3.1.74)$$

becomes,

$$-1250 < 0.00383p + 0.804t < 1500 \qquad (7.3.1.75)$$

This inequality defines a region in the p-t coordinate system, that is, p-t space. The region defines the values of tension and torsion that can simultaneously be applied to the tube and not cause the +20° layers to fail. Of course, for no torsion equation (7.3.1.9) is recovered. As in previous discussions, it is the boundaries of the region that are important, and as in the previous examples, it is the equalities associated with the inequalities that define the boundaries. Specifically, in the present situation, the first equation of equation (7.3.1.4), namely, the compression side of the inequality,

$$\sigma_1 = \sigma_1^C \qquad (7.3.1.79)$$

results in,

$$0.00383p + 0.804t = -1250 \qquad (7.3.1.77)$$

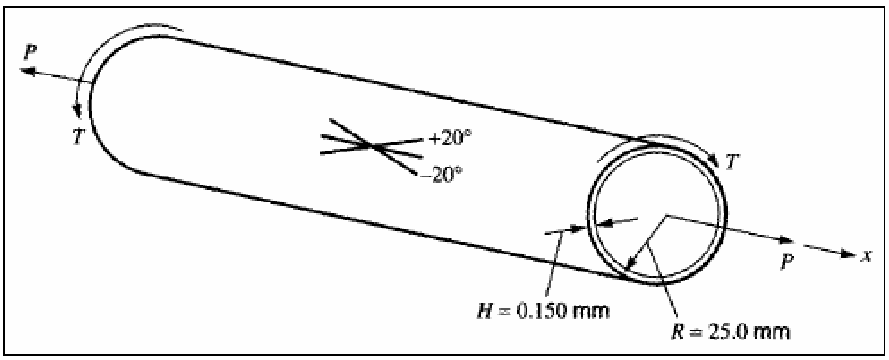


Figure M7.3.1.10 Tube with axial load P and torsion T, Failure Example 3

Layer	σ_1	σ_2	τ_{12}
+20°	$+0.00383p + 0.804t$	$-0.0001123p - 0.0481t$	$-0.0001487p + 0.0552t$
–20°	$+0.00383p - 0.804t$	$-0.0001123p + 0.0481t$	$+0.0001487p + 0.0552t$
0°	$+0.00477p$	$-0.0001686p$	$+0.0721t$

Table M7.3.1.12 Stresses (MPa) in [±20/03]s tube loaded with P= p (N) and T= t (N m)

Unlike the past examples where enforcement of the equalities led to a specific value of p or t, here enforcement of the equality leads to an equation for a line in p-t space. This line, illustrated in Figure M7.3.1.11, is labeled σ_1^C, the notation identifying the failure mode represented by this line. The line divides p-t space into two regions, where any combination of p and t on this line and below it represents a combination that will cause a compression failure in the fiber direction of the +20° layers. A combination of p and t above this line represents values that will be safe from causing compression failure in the fiber direction of the +20° layers.

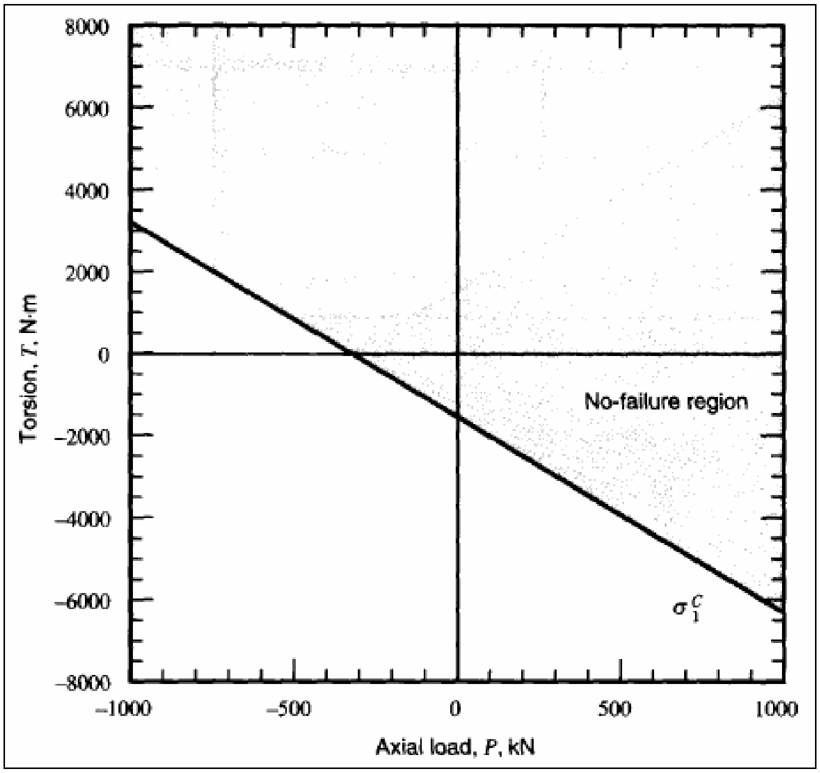


Figure M7.3.1.11 Failure boundaries for compression failure in 1 direction in +20° layers

The equation from the tension side of the inequality is,

$$\sigma_1 = \sigma_1^T \qquad (7.3.1.78)$$

or

$$0.00383p + 0.804t = 1500 \qquad (7.3.1.79)$$

This represents another line in p-t space, illustrated in Figure M7.3.1.12, where the notation σ_1^T is assigned to the line represented by equation (7.3.1.79) to denote that the line represents failure due to tension in the fiber direction. The σ_1^T line again divides p-t space into two parts. In this case, the region on the line and above it represents the region where any combination of p and t will lead to failure of the +20° layers because of tension in the fiber direction, and the region below the line represents combinations of loads that will not cause the +20° layers to fail in tension in the fiber direction. Obviously the region between the σ_1^C line of Figure M7.3.1.11 and the a\ line of Figure M7.3.1.12 represents combinations of p and t in p-t space that will not cause the +20° layers to fail in the fiber direction. The two lines are shown in Figure M7.3.1.13; the region free from fiber-direction failure is shaded in and denoted as safe.

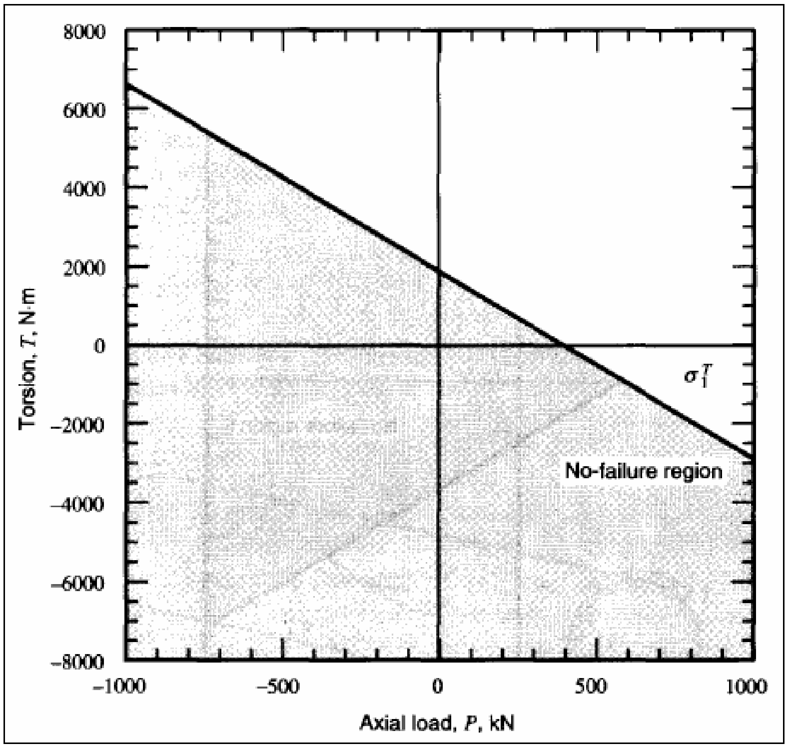


Figure M7.3.1.12 Failure boundary for tension failure in 1 direction in +20° layers

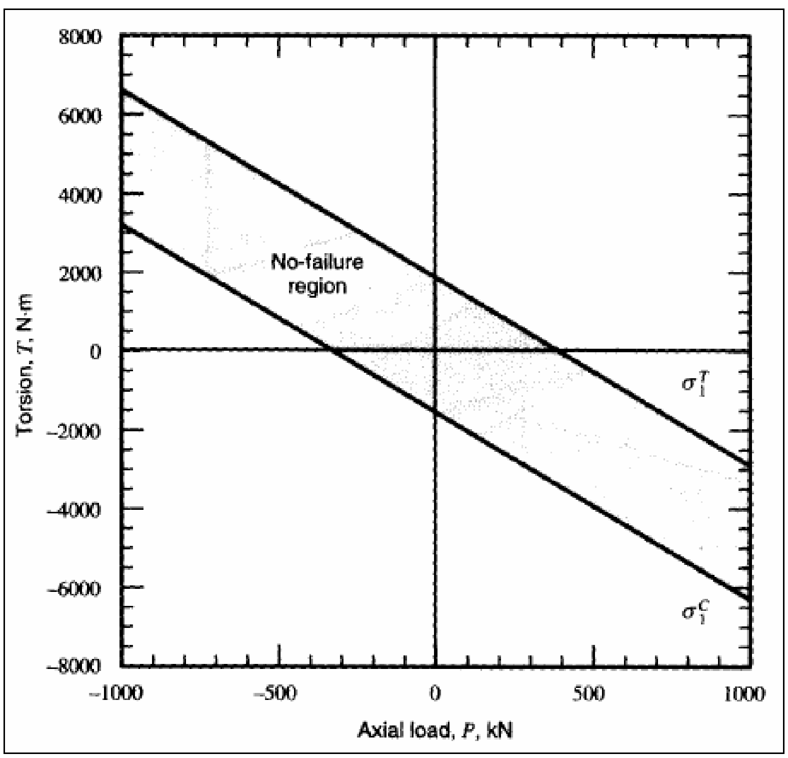


Figure M7.3.1.13 Failure envelope for failure in 1 direction in +20° layers

From equation (7.3.1.3), the equation that represents failure in the 2 direction,

$$\sigma_2^C < \sigma_2 < \sigma_2^T \qquad (7.3.1.80)$$

leads to,

$$-200 < -0.0001123p - 0.0481t < 50 \qquad (7.3.1.81)$$

This results in two equations for two more boundary lines in p-t space. From the compression portion,

$$-0.0001123p - 0.0481t = -200 \qquad (7.3.1.82)$$

and from the tension portion,

$$-0.0001123p - 0.0481t = 50 \qquad (7.3.1.83)$$

In Figure M7.3.1.14 the lines represented by these two equations are added to the lines of Figure M7.3.1.13 and the notation σ_2^C and σ_2^T is used to identify these lines. As we can see, the boundary associated with compression failure in the 2 direction is considerably removed from the previously established boundaries associated with failure in the fiber direction. This is interpreted to mean that failure due to compression in the 2 direction is not possible with any combination of p and t before failure occurs in the fiber direction due to tension. On the other hand, the line representing tension failure in the 2 direction intersects the previously established region for failure in the 1 direction. Thus, the safe region is impinged upon by failure in another mode and the size of the safe region is reduced.

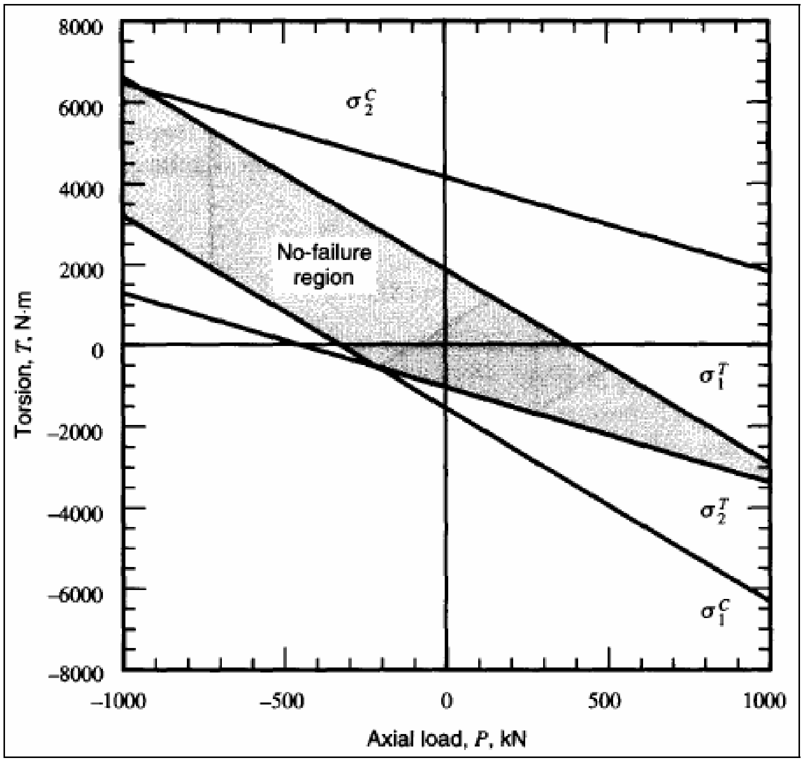


Figure M7.3.1.14 Failure envelope for failure in 1 and 2 directions in +20° layers

Finally, failure in the +20° layers due to shear stress τ_{12} is given by the third equation of equation (7.3.1.3), namely,

$$-\tau_1^F < \tau_{12} < \tau_{12}^F \quad (7.3.1.84)$$

which, for the case here, becomes,

$$-100 < -0.0001487p + 0.0552t < 100 \quad (7.3.1.85)$$

This inequality defines yet a third region in p-t space, whose regional boundaries are given by,

$$-0.0001487p + 0.0552t = -100 \quad (7.3.1.86)$$

and,

$$-0.0001487p + 0.0552t = +100 \quad (7.3.1.87)$$

In Figure M7.3.1.15 the lines represented by these equations are added to Figure M7.3.1.14. Both of these new lines intersect the previously established safe region and further restrict its size. The irregular-shaped shaded region bounded by segments of the various lines corresponding to the six failure equalities represents the range of values of axial load and torsional load that can be applied to the tube without having the +20° layers fail in any of the various modes. The shape of the region, the intersections of the lines with the coordinate axes, the intersections of the lines with each other, and other characteristics of the figure are a result of the material elastic properties, the fiber angles in the various layers, and the failure stresses of the material. Actually the tube radius is reflected in the characteristics of the region. Hence, Figure M7.3.1.15 summarizes a great deal of information regarding this particular problem. Also, on Figure M7.3.1.15 the points for $T = 0$, which correspond to the first example problem, are noted, as are the points for P = 225 kN, which correspond to the second example.

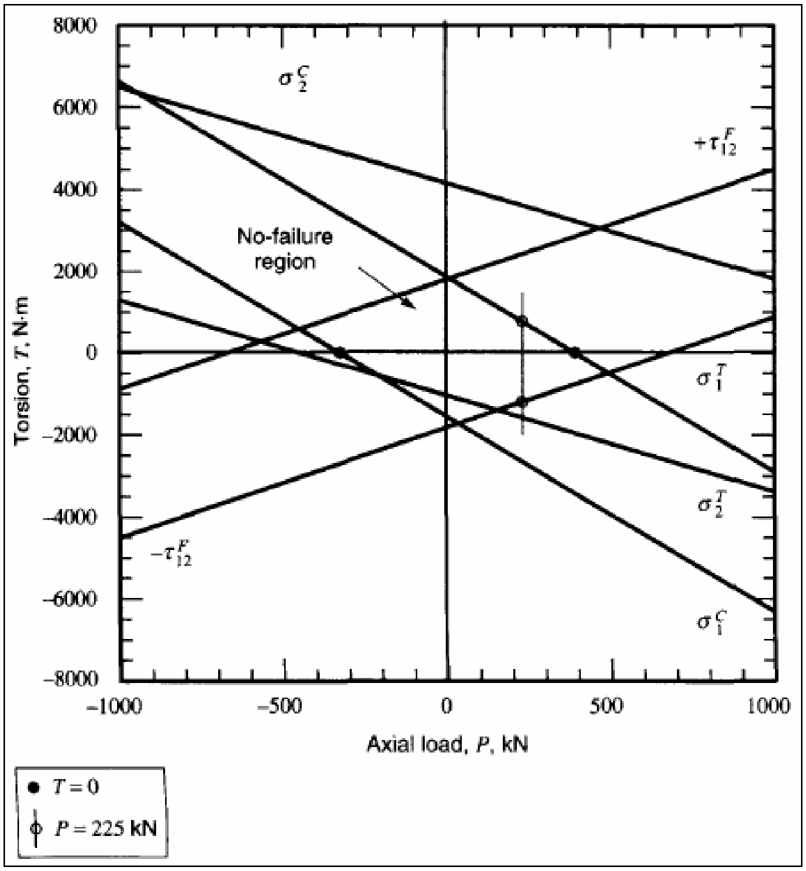


Figure M7.3.1.15 Complete failure envelope for +20° layers

With the failure boundaries established for the +20° layers, attention turns to the —20° and 0° layers. The determination of the failure boundaries for these layers follows steps identical to those just completed for the —20° layers. For the —20° layers the analysis is as follows:

Compression failure in the 1 direction:

$$0.00383p - 0.804t = -1250 \quad (7.3.1.88)$$

Tension failure in the 1 direction:

$$0.00383p - 0.804t = 1500 \quad (7.3.1.89)$$

Compression failure in the 2 direction:

$$-0.0001123p + 0.0481t = -200 \qquad (7.3.1.90)$$

Tension failure in the 2 direction:

$$-0.0001123p + 0.0481t = 50 \qquad (7.3.1.91)$$

Shear failure:

$$0.0001487p + 0.0552t = -100 \qquad (7.3.1.92)$$

and

$$0.0001487p + 0.0552t = 100 \qquad (7.3.1.93)$$

Figure M7.3.1.16 illustrates the six lines associated with the failure boundaries and the region in p-t space free from failure for the —20° layers. Note the differences and similarities between Figure M7.3.1.15 for the +20° layers and Figure M7.3.1.16. Whereas the shaded region for the +20° layers is skewed downward slightly on the right, the region for the —20° layers is skewed upward slightly on the right. The points corresponding to the two previous failure examples are indicated.

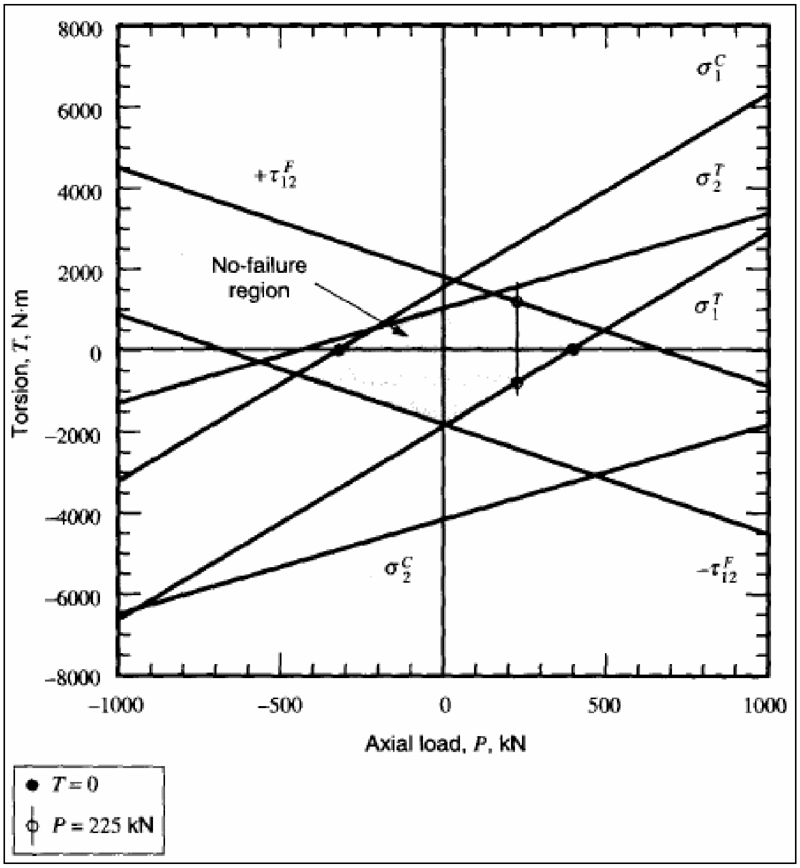


Figure M7.3.1.16 Complete failure envelope for —20° layers

For the 0° layers: Compression failure in the 1 direction:

$$0.00477p = -1250 \tag{7.3.1.94}$$

Tension failure in the 1 direction:

$$0.00477p = 1500 \tag{7.3.1.95}$$

Compression failure in the 2 direction:

$$-0.0001686p = -200 \tag{7.3.1.96}$$

Tension failure in the 2 direction:

$$-0.0001686p = 50 \tag{7.3.1.97}$$

Shear failure:

$$0.0721t = -100 \tag{7.3.1.98}$$

and

$$0.0721t = +100 \qquad (7.3.1.99)$$

As shown in Figure M7.3.1.17, the lines representing failure of the 0° layers are all parallel to the coordinate axes in p-t space. The line representing compression failure in the 2 direction is not shown since it is far to the right. The region free from failure is a rectangular region bounded on the top and bottom by shear failure, and bounded on the left and right by failure in the fiber direction.

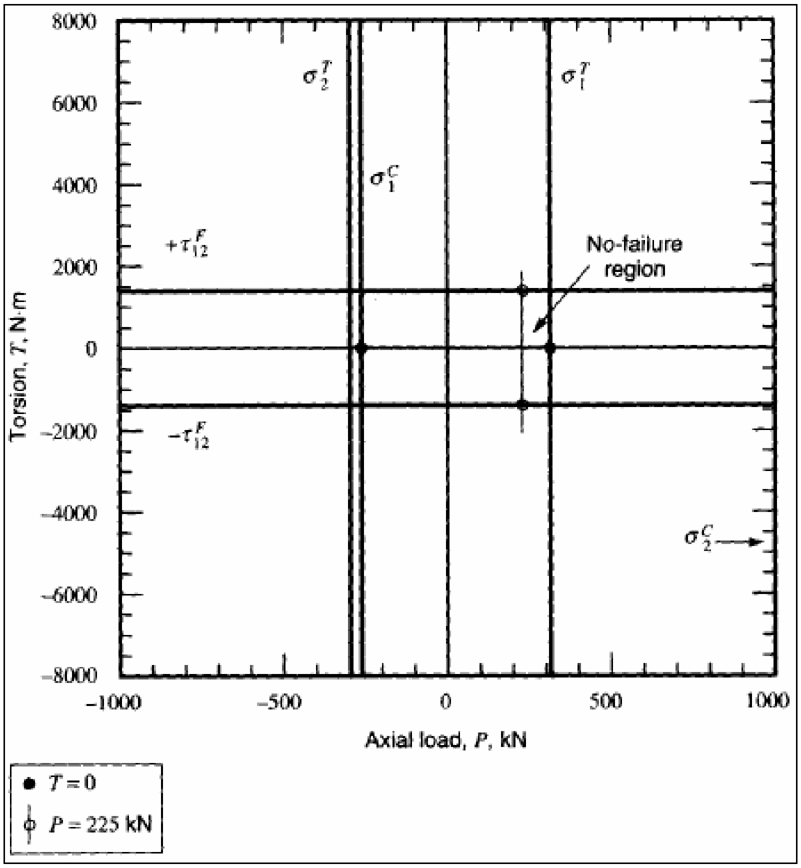


Figure M7.3.1.17 Complete failure envelope for 0° layers

Finally, the superposition of the boundaries that control failure of die laminate is shown in Figure M7.3.1.18, where the boundaries are labeled as to the failure mode and the layer orientation that controls failure. In general, stress components σ_1 and σ_2 in the ±20° layers control failure. However, at extreme values of p, failure in the fiber direction in the 0° layers controls failure. Figure M7.3.1.18 is important because from it one can determine the level of torque that can be applied for a specific level of axial load.

We mentioned earlier that the analyses being presented were applicable to first-ply failure loads, and that there might be additional load capacity beyond the load at which the first failure occurs. This may be true particularly when the first failure does not involve fiber failure as when, for example, the first failure is due to tension in the 2 direction. For this case the matrix could develop cracks parallel to the fibers, but there would still be integrity to the fibers. To reflect the fact that matrix cracking has occurred in specific layers at a particular load level, E_2 and G_{12} in those layers could be reduced significantly or equated to zero, and the A matrix recomputed. A failure analysis could then be conducted on the altered laminate, and a new failure load could be predicted. This progressive recomputation of the A matrix (and the B and D matrices, if they are involved) can continue, and ultimate failure can be assumed to occur when stresses in the fiber direction in the layer most highly stressed in the fiber direction exceed the failure stress level.

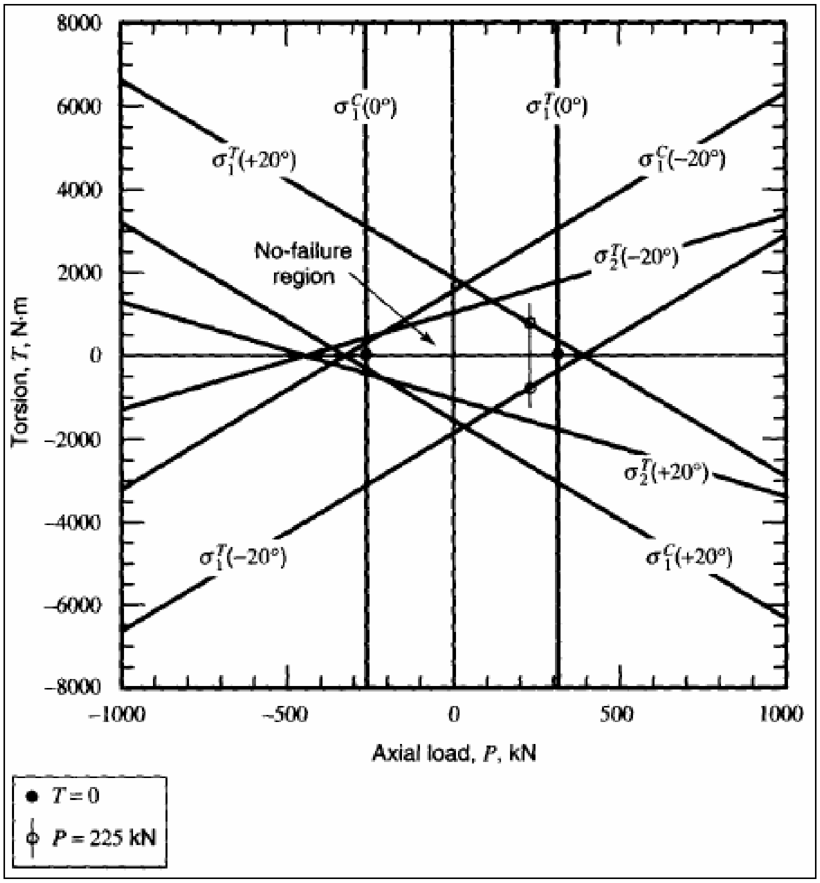


Figure M7.3.1.18 Superposition of failure envelopes for +20°, —20°, and 0° layers

Exercise for Section 7.3.1.4

A [±45/02] s graphite-reinforced plate is subjected to a biaxial loading; that is:

$$N_x = N_x$$
$$N_y = N_y$$
$$N_{xy} = 0$$

Use the maximum stress failure criterion to determine the failure envelope of the plate in $N_x - N_y$ space. Label the failure mode and layer or layers which control failure for each portion of the envelope. Note that the cases of Exercise 2 in the Exercises for Section 7.3.1.2 and the Exercise for Section 7.3.1.3 are included in this case.

We shall now return to our previous examples with the flat $[0/90]_S$ and $[0/30]_S$ laminates and study failure for those cases. In particular, we shall study failure for the situation where the stress resultants, rather than the deformations, are specified. Because we have studied these examples in some depth, using them to study failure will provide us with additional insight into the response of fiber-reinforced laminated composites. Also, because we have studied these laminates with particular loadings, we have figures illustrating the stress distributions within the layers. With the stress distributions available, it might be tempting to focus only on the layer or layers with the highest stresses. Unfortunately, with the dramatically different failure strengths in the different directions in a layer, and in tension and compression, a focusing a priori on one layer or one failure mode in one layer can lead to the wrong conclusions. A failure mode may be overlooked, or the closeness of the loads for different failure modes in different layers may not be noted. Thus, we shall avoid the temptation to focus on one layer or one failure mode despite what may appear as overwhelming evidence that failure will occur in a particular way. Rather, we will use the same methodical approach to study failure that we have used in the tube examples. As this can all be automated by computer programming, there is every reason to take the safe and thorough approach to the study of failure.

Summary

As we can see, implementing the maximum stress failure criterion requires a number of calculations to determine the load level that causes failure. The different modes of failure must be checked, and each layer considered. If bending is present, positive and negative z locations must be checked, though the computation need only be done for positive z for many laminates. These steps can all be automated and really should be when applying the criterion routinely. For combined loading, graphical displays of the failure envelopes are very useful.

In the implementation of the maximum stress failure criterion, the individual failure modes were examined one at a time. When computing the load required to failing the fibers, for example, we found that there was no concern for the magnitude, or direction, of the stress perpendicular to the fibers σ_2. That is, there was no concern that σ_2 might interact with σ_1, so the failure load predicted by accounting for the presence of σ_2 might be different from the failure load predicted by ignoring the presence of σ_2. The concept of considering more than one stress component at a time when studying failure is termed stress interaction. The next chapter examines a failure criterion that addresses this issue.

M7.3.2 Failure Theories for Fiber-Reinforced Materials: Maximum Strain Theory

Failure occurs when at least one of the strain components along the principal material axis exceeds that the ultimate strain in that direction.

Tensile strain:

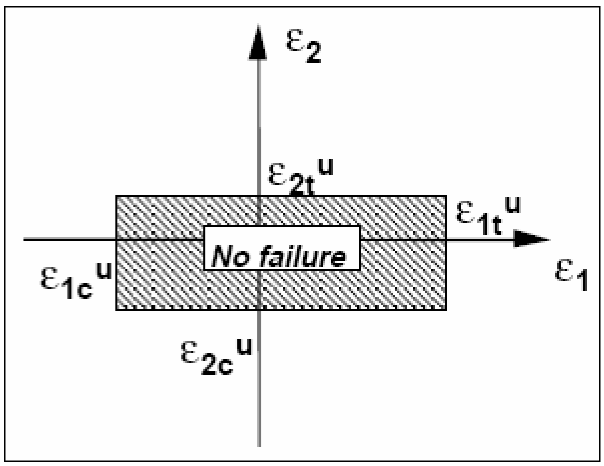
$$\varepsilon_1 \geq \varepsilon_{1t}^u$$
$$\varepsilon_2 \geq \varepsilon_{2t}^u$$

Compressive strain:

$$\varepsilon_1 \leq \varepsilon_{1c}^u$$
$$\varepsilon_2 \leq \varepsilon_{2c}^u$$

Shear strain:

$$|\gamma_{12}| \geq \gamma_6^u \qquad or \quad |\gamma_6| \geq \gamma_6^u$$



M7.3.2.1 Maximum Strain Theory Expressed in Stresses

Maximum strains:

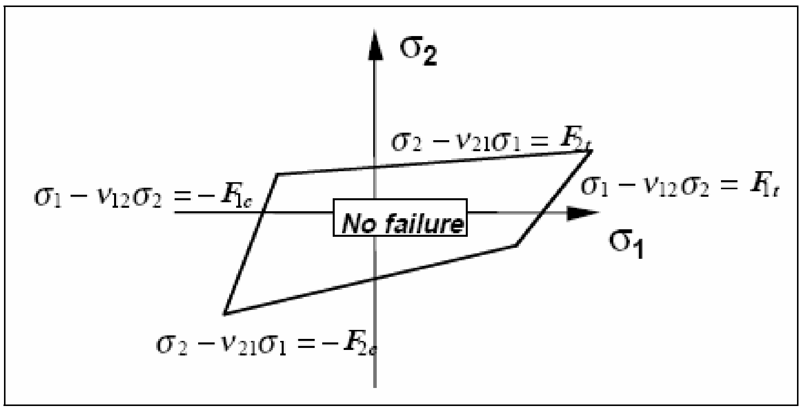
$$\varepsilon_1 = (\sigma_1 - \nu_{12}\sigma_2)/E_1$$
$$\varepsilon_2 = (\sigma_2 - \nu_{21}\sigma_1)/E_2$$
$$\gamma_6 = \tau_6/G_{12}$$

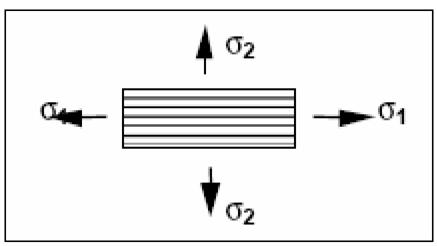
At failure:

$$\varepsilon_1 = \varepsilon^u_{1t} \quad or \quad -\varepsilon^u_{1c}$$
$$\varepsilon_2 = \varepsilon^u_{2t} \quad or \quad -\varepsilon^u_{2c}$$
$$|\gamma_6| = \gamma^u_6$$

Ultimate strains are calculated from Uniaxial & Shear tests:

$$\varepsilon^u_{1t} = \frac{F_{1t}}{E_1} \quad and \quad \varepsilon^u_{1c} = \frac{F_{1c}}{E_1}$$
$$\varepsilon^u_{2t} = \frac{F_{2t}}{E_2} \quad and \quad \varepsilon^u_{2c} = \frac{F_{2c}}{E_2}$$
$$\gamma^u_6 = F_6 / G_{12}$$





M7.3.2.2 Application of Maximum Strain Theory to Angle-ply Laminate

Strains:

$$\varepsilon_1 = (\sigma_1 - \nu_{12}\sigma_2)/E_1$$
$$\varepsilon_2 = (-\nu_{21}\sigma_1 + \sigma_2)/E_2$$

Tension Loaded:

$$\sigma_x = \frac{F_{1t}}{Cos^2\theta - \nu_{12}Sin^2\theta}$$

$$\sigma_x = \frac{F_{2t}}{Sin^2\theta - \nu_{21}Cos^2\theta}$$

Compression Loaded:

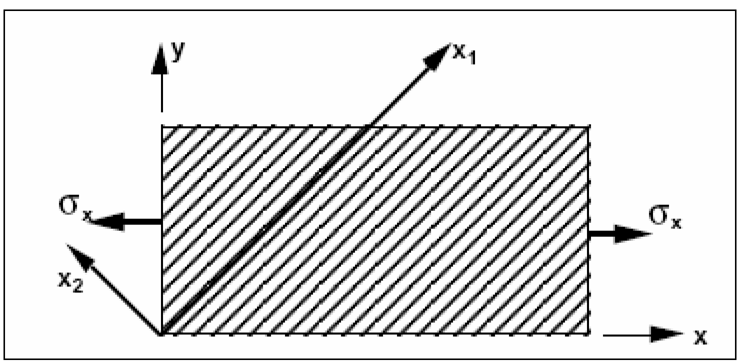
$$\sigma_x = -\frac{F_{1c}}{Cos^2\theta - \nu_{12}Sin^2\theta} \quad \Rightarrow Longitudinal$$

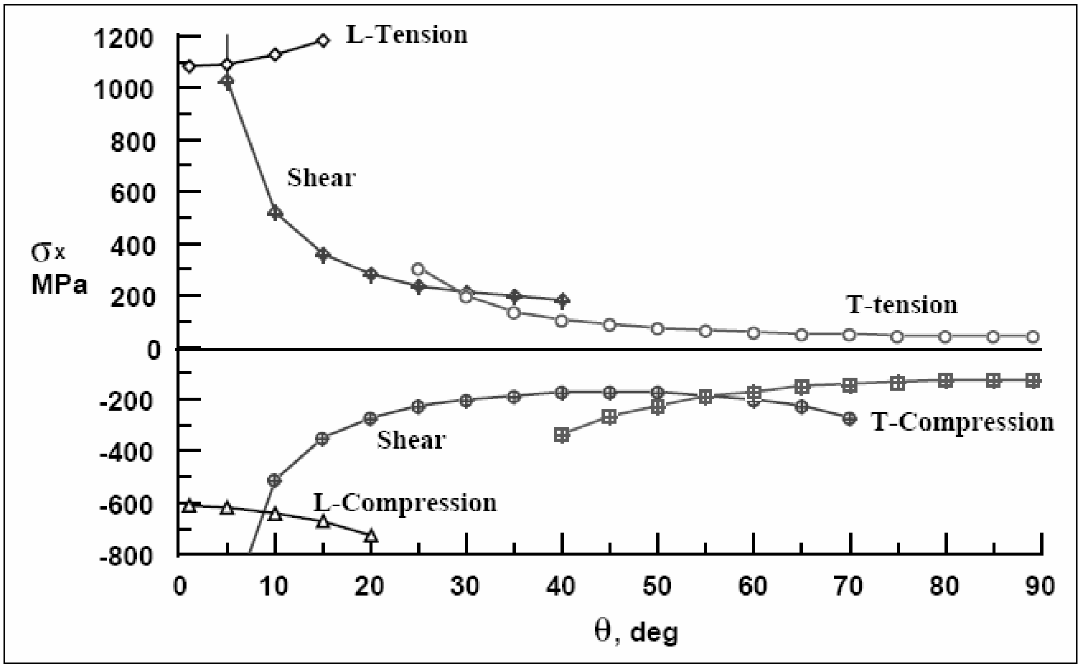
$$\sigma_x = -\frac{F_{2c}}{Sin^2\theta - \nu_{21}Cos^2\theta} \quad \Rightarrow Transverse$$

Shear Loaded:

$$\sigma_x = \pm\frac{F_6}{Cos\theta Sin\theta} \quad \Rightarrow Shear$$

M7.3.2.3 Maximum Strain Theory: Uniaxial Strength of an Off-Axis Lamina





M7.3.3 Failure Theories for Fiber-Reinforced Materials: Tsai-Hill Theory

Hill extended the von Mises criterion for ductile anisotropic material. Azzi-Tsai extended this equation to anisotropic fiber reinforced composites. Failure occurs when the LHS of the following equation is equal to or greater than one.

$$A\sigma_1^2 + B\sigma_2^2 + C\sigma_1\sigma_2 + D\tau_6^2 = 1$$

From longitudinal, transverse, and shear tests on a uniaxial laminate, A, B, and D are determined.

$$A = \frac{1}{F_1^2}, \quad B = \frac{1}{F_2^2}, \quad and \quad D = \frac{1}{F_6^2}$$

From Equal Biaxial test:

Failure occurs when the transverse stress (F_2) reaches F_2.

$$C_1 = -1/F_1^2$$

Tsai-Hill failure criterion:

$$\frac{\sigma_1^2}{F_1^2} + \frac{\sigma_2^2}{F_2^2} - \frac{\sigma_1\sigma_2}{F_1^2} + \frac{\tau_6^2}{F_6^2} = 1$$

$$\frac{\sigma_1^2}{F_1^2} + \frac{\sigma_2^2}{F_2^2} - \frac{\sigma_1\sigma_2}{F_1^2} = 1 - \kappa^2$$

$$\kappa = \frac{\tau_6}{F_6}$$

Note: No distinction is made between tensile & compression strengths.

7.3.3.1 Application of Tsai-Hill Failure Criterion to Angle-Ply Laminate

Substitute for σ_1, σ_2**, and** τ_{12} **in terms of** σ_x **in:**

$$\frac{\sigma_1^2}{F_1^2} + \frac{\sigma_2^2}{F_2^2} - \frac{\sigma_1\sigma_2}{F_1^2} + \frac{\tau_6^2}{F_6^2} = 1$$

We get the failure stress:

$$\frac{1}{\sigma_x^2} = \frac{Cos^4\theta}{F_{1t}^2} + \frac{Sin^4\theta}{F_{2t}^2} + \left(\frac{1}{F_6^2} - \frac{1}{F_{1t}^2}\right) Cos^2\theta Sin^2\theta \qquad \textit{For Tensile Stresses}$$

$$\frac{1}{\sigma_x^2} = \frac{Cos^4\theta}{F_{1c}^2} + \frac{Sin^4\theta}{F_{2c}^2} + \left(\frac{1}{F_6^2} - \frac{1}{F_{1c}^2}\right) Cos^2\theta Sin^2\theta \qquad \textit{For Compressive Stresses}$$

M7.3.4 Failure Theories for Fiber-Reinforced Materials: The Tsai-Wu Criterion

The von Mises criterion, introduced in strength-of-materials courses for studying yielding of metals, can be written as

$$\frac{1}{2}\left(\frac{1}{\sigma^Y}\right)^2 \left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right] = 1 \qquad (7.3.4.1)$$

where σ^Y is the yield stress of the metal and σ_1, σ_2, and σ_3 are the principal stresses. This equation is of the form,

$$F(\sigma_1, \sigma_2, \sigma_3) = 1 \qquad (7.3.4.2)$$

According to the von Mises criterion, if,

$$F(\sigma_1, \sigma_2, \sigma_3) < 1 \qquad (7.3.4.3)$$

then the material has not yielded. Equation (7.3.4.1) represents the well-known von Mises ellipsoid and thus a surface in $\sigma_1 - \sigma_2 - \sigma_3$, principal stress space, and equation (7.3.4.3) represents the volume inside this surface. Because rolled metals have slightly different properties in the roll direction than in the other two perpendicular directions, Hill (see Suggested Readings) assumed that the yield criterion for these orthotropic metals was of the form,

$$F(\sigma_1 - \sigma_2)^2 + G\,(\sigma_1 - \sigma_3)^2 + H\,(\sigma_2 - \sigma_3)^2 + 2L\tau_{12}^2 + 2M\tau_{13}^2 + 2N\tau_{23}^2 = 1 \qquad (7.3.4.4)$$

The constants F, G, H, and so forth, are related to the yield stresses in the different directions, like σ^Y in equation (7.3.4.1), and either the 1, 2, or 3 direction is aligned with the roll direction.

This view of a failure criterion can be extended to composite materials, which are, of course, orthotropic in the principal material coordinate system, by assuming an equation of the form,

$$F(\sigma_1, \sigma_2, \sigma_3, \tau_{23}, \tau_{13}, \tau_{12}) = 1 \qquad (7.3.4.5)$$

can be used to represent the failure condition of a composite, while the condition of no failure is given by,

$$F(\sigma_1, \sigma_2, \sigma_3, \tau_{23}, \tau_{13}, \tau_{12}) < 1 \qquad (7.3.4.6)$$

How do we determine the specific form of F for a composite? How do we know such a function can exist? Is the concept even valid? After all, composites are not like metals when it comes to failure. There is distinctly different failure mechanisms associated with failure of a composite: fiber kinking in compression, fiber fracture in tension, and failure at the fiber-matrix interface in shear or tension perpendicular to the fiber. Thus, why would extending the concepts from metals be valid for composites? In the strictest sense, it is not. However, if the generalization is viewed as a hypothesis for fitting empirical data, and if the fit is reasonable, then the hypothesis provides us with an indicator that can be used to study failure.

For a state of plane stress, if the power of the stress components is maintained at 2, as in equations (7.3.4.1) and (7.3.4.4), the most general form of F is

$$F(\sigma_1, \sigma_2, \tau_{12}) = F_1\sigma_1 + F_2\sigma_2 + F_6\tau_{12} + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2 + 2F_{16}\sigma_1\tau_{12} + 2F_{26}\sigma_2\tau_{12} \quad (7.3.4.7)$$

where in the above $F_1, F_2, F_6, F_{11}, F_{22}, F_{66}, F_{12}, F_{16}$, and F_{26} are constants. All stress components are represented to the first and second powers, and all products of the stresses are represented. The constants F_{12}, F_{16}, and F_{26} will be referred to as the interaction constants, and the magnitude of their value will dictate the degree of interaction among stress components. Interaction between the normal stresses σ_1 and σ_2 and the shear stress τ_{12} is included by virtue of constants F_{16}, and F_{26}, and interaction between the normal stress components σ_1 and σ_2 is included with the F_{12} term.

With the above considerations, the failure criterion that we are seeking takes the form

$$F_1\sigma_1 + F_2\sigma_2 + F_6\tau_{12} + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2 + 2F_{16}\sigma_1\tau_{12} + 2F_{26}\sigma_2\tau_{12} = 1 \quad (7.3.4.8)$$

and the condition of no failure is given by the inequality

$$F_1\sigma_1 + F_2\sigma_2 + F_6\tau_{12} + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2 + 2F_{16}\sigma_1\tau_{12} + 2F_{26}\sigma_2\tau_{12} < 1 \quad (7.3.4.9)$$

The failure criterion represents a general second-order surface in the space with co-ordinates $\sigma_1, \sigma_2, \tau_{12}$. Recall, the maximum stress criterion in Figure M7.3.1.7 represented a piecewise planar surface with sharp edges and corners. A second-order surface, on the other hand, is smooth, like an ellipsoid. If we know the values of constants $F_1, ..., F_{66}$, then we can construct the surface, if desired, and furthermore, the failure load of a laminate can be evaluated by using equation (7.3.4.8). Equation (7.3.4.8) is the plane-stress form of the failure criterion postulated by Tsai and Wu (see Suggested Readings). How do we determine the constants $F_1, ..., F_{66}$? The answer is simple! We evaluate them with the information we have regarding failure of an element of fiber-reinforced material, namely: we evaluate them in terms of $\sigma_1^T, \sigma_1^C, \sigma_2^T, \sigma_2^C$ and τ_{12}^F.

M7.3.4.1 Determination of the Constants

We determine the constants $F_1, ..., F_{66}$ by referring to the results of simple failure tests with fiber-reinforced composite material. Consider an element of material subjected to a stress only in the fiber direction. For this situation

$$\sigma_1 \neq 0$$
$$\sigma_2 = 0$$
$$\tau_{12} = 0 \tag{7.3.4.10}$$

The function $F(\sigma_1, \sigma_2, \tau_{12})$ of equation (7.3.4.7) reduces to the form

$$F(\sigma_1, \sigma_2, \tau_{12}) = F_1\sigma_1 + F_{11}\sigma_1^2 \tag{7.3.4.11}$$

If, as in Figure M7.3.4.1, the stress σ_1 is tension, then failure occurs when $\sigma_1 = \sigma_1^T$. The failure criterion must be unity at this value of σ_1, that is:

$$F_1\sigma_1^T + F_{11}\left(\sigma_1^T\right)^2 = 1 \tag{7.3.4.12}$$

If, on the other hand, the stress σ_1 is compression, as in Figure M7.3.4.2, then failure occurs when $\sigma_1 = \sigma_1^C$. The failure criterion must again be unity at this value of σ_1, namely,

$$F_1\sigma_1^C + F_{11}\left(\sigma_1^C\right)^2 = 1 \tag{7.3.4.13}$$

Equations (7.3.4.12) and (7.3.4.13) resulting from the two tests for failure in the fiber direction provide enough information to solve for F_1 and F_{11}. These two equations result in

$$F_1 = \frac{1}{\sigma_1^T} + \frac{1}{\sigma_1^C} \qquad F_{11} = -\frac{1}{\sigma_1^T\sigma_1^C} \tag{7.3.4.14}$$

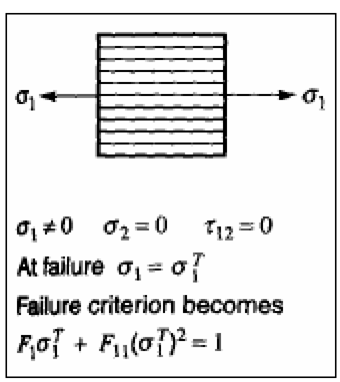


Figure M7.3.4.1 Tensile failure in 1 direction as it applies to the Tsai-Wu criterion

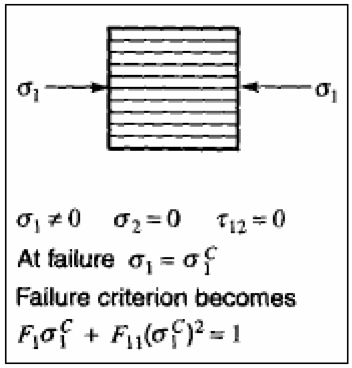


Figure M7.3.4.2 Compression failure in 1 direction as it applies to the Tsai-Wu criterion

We can take a similar approach with tension and compression testing of an element of material in the 2 direction. For this situation

$$\sigma_1 = 0$$
$$\sigma_2 \neq 0$$
$$\tau_{12} = 0 \qquad (7.3.4.15)$$

The function $F(\sigma_1, \sigma_2, \tau_{12})$ of equation (7.3.4.7) reduces to the form

$$F(\sigma_1, \sigma_2, \tau_{12}) = F_2\sigma_2 + F_{22}\sigma_2^2 \qquad (7.3.4.16)$$

If the stress σ_2 is tension, then failure occurs when $\sigma_1 = \sigma_1^T$ and the failure criterion becomes

$$F_2\sigma_2^T + F_{22}\left(\sigma_2^T\right)^2 = 1 \qquad (7.3.4.17)$$

If the stress σ_2 is compression, then failure occurs when $\sigma_1 = \sigma_1^C$, resulting in

$$F_2\sigma_2^C + F_{22}\left(\sigma_2^C\right)^2 = 1 \qquad (7.3.4.18)$$

From equations (7.3.4.17) and (7.3.4.18),

$$F_2 = \frac{1}{\sigma_2^T} + \frac{1}{\sigma_2^C} \qquad F_{22} = -\frac{1}{\sigma_2^T\sigma_2^C} \qquad (7.3.4.19)$$

These two loading conditions with σ_2 are shown in Figures 7.3.4.3 and 7.3.4.4.

The results from testing to failure in shear an element of material can be used to determine two more constants in the criterion. If an element is loaded only in shear, then

$$\sigma_1 = 0$$
$$\sigma_2 = 0$$
$$\tau_{12} \neq 0 \qquad (7.3.4.20)$$

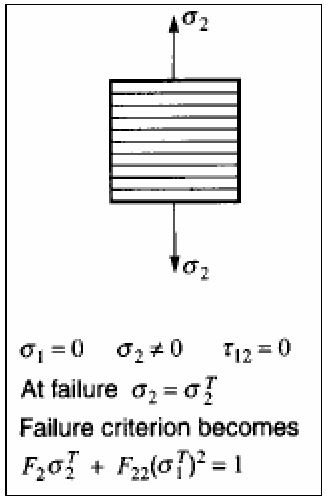


Figure M7.3.4.3 Tension failure in 2 directions as it applies to the Tsai-Wu criterion

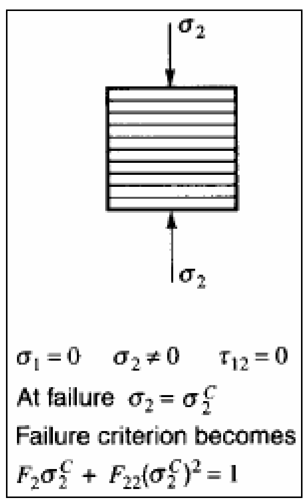


Figure M7.3.4.4 Compression failure in 2 directions as it applies to the Tsai-Wu criterion

The function $F(\sigma_1, \sigma_2, \tau_{12})$ of equation (7.3.4.7) becomes

$$F(\sigma_1, \sigma_2, \tau_{12}) = F_6\tau_{12} + F_{66}\tau_{12}^2 \qquad (7.3.4.21)$$

If the stress τ_{12} is positive, as shown in Figure M7.3.4.5, then failure occurs when $\tau_{12} = \tau_{12}^F$ and

$$F_6\tau_{12}^F + F_{66}\left(\tau_{12}^F\right)^2 = 1 \tag{7.3.4.22}$$

If the stress τ_{12} is reversed, as shown in Figure M7.3.4.6, then failure occurs when

$$-F_6\tau_{12}^F + F_{66}\left(-\tau_{12}^F\right)^2 = 1 \tag{7.3.4.23}$$

There two equations lead to,

$$F_6 = 0 \qquad F_{66} = \left(\frac{1}{\tau_{12}^F}\right)^2 \tag{7.3.4.24}$$

Basically, F_6 is zero because in the principal material coordinate system failure is not sensitive to the sign of the shear stress. This certainly makes physical sense and extends to nonplanar stress situations as well. As a result, the failure criterion only involves the shear stress squared, reflecting the insensitivity to sign.

To this point, the Tsai-Wu failure criterion, equation (7.3.4.8), becomes

$$\left(\frac{1}{\sigma_1^T} + \frac{1}{\sigma_1^C}\right)\sigma_1 + \left(\frac{1}{\sigma_2^T} + \frac{1}{\sigma_2^C}\right)\sigma_2 + \left(-\frac{1}{\sigma_1^T\sigma_1^C}\right)\sigma_1^2 + \left(-\frac{1}{\sigma_2^T\sigma_2^C}\right)\sigma_2^2 + \left(\frac{1}{\tau_{12}^F}\right)^2\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2 + 2F_{16}\sigma_1\tau_{12} + 2F_{26}\sigma_2\tau_{12} = 1 \tag{7.3.4.25}$$

We need to evaluate coefficients that involve the product of two stress components. Whereas we could evaluate F_1 and F_{11} by looking at the results of testing to failure an element of material with a single stress component applied, namely, σ_1, to determine F_{12}, F_{16}, and F_{26} the failure of an element of material must be studied when stressed by more than one component. This form of testing is difficult and it can be expensive. However, two of the three remaining coefficients can be shown to be zero on physical grounds. The third coefficient can be estimated without resorting to actual experimental results.

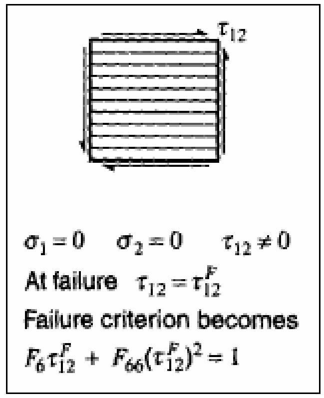


Figure M7.3.4.5 Failure due to positive τ_{12} as it applies to the Tsai-Wu criterion

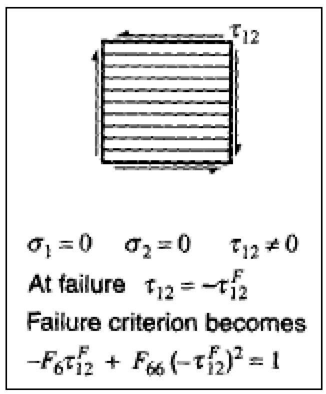


Figure M7.3.4.6 Failure due to negative τ_{12} as it applies to the Tsai-Wu criterion

To determine F_{16}, consider an element of fiber-reinforced material subjected to a tensile stress in the fiber direction. Assume, as in Figure M7.3.4.7(a), the tensile stress has a specific and known value σ_1^*. Now suppose a shear stress τ_{12} is superposed on the element. Assume the shear stress is started from zero and increased until the element fails. Suppose, as Figure M7.3.4.7(b) shows, the value of τ_{12} that causes failure is τ_{12}^*. Because the element is stressed to failure, the failure criterion, equation (7.3.4.8), can be written as

$$F_1\sigma_1^* + F_{11}\left(\sigma_1^*\right)^2 + F_{66}\left(\tau_{12}^*\right)^2 + 2F_{16}\sigma_1^*\tau_{12}^* = 1 \qquad (7.3.4.26)$$

Recall that F_1, F_{11}, and F_{66} are known, and F_6 was shown to be zero. Now consider another element of material also loaded in the fiber direction by a tensile stress $\sigma_1 = \sigma_1^*$. A negative shear stress τ_{12} is applied to this element and increased in magnitude until the element fails. Because this experiment is being conducted in the principal material system, it is logical to assume, as in Figure M7.3.4.7(c), that the value of shear stress that causes failure is $\tau_{12} = -\tau_{12}^*$. If this is the case, then the failure criterion can be written as

$$F_1\sigma_1^* + F_{11}\left(\sigma_1^*\right)^2 + F_{66}\left(\tau_{12}^*\right)^2 - 2F_{16}\sigma_1^*\tau_{12}^* = 1 \qquad (7.3.4.27)$$

If equation (7.3.4.27) is subtracted from equation (7.3.4.26), the result is

$$4F_{16}\left(\tau_{12}^*\right)^2 = 0 \qquad (7.3.4.28)$$

from which we can conclude that

$$F_{16} = 0 \qquad (7.3.4.29)$$

A similar argument can be made regarding F_{26}, that is:

$$F_{26} = 0 \qquad (7.3.4.30)$$

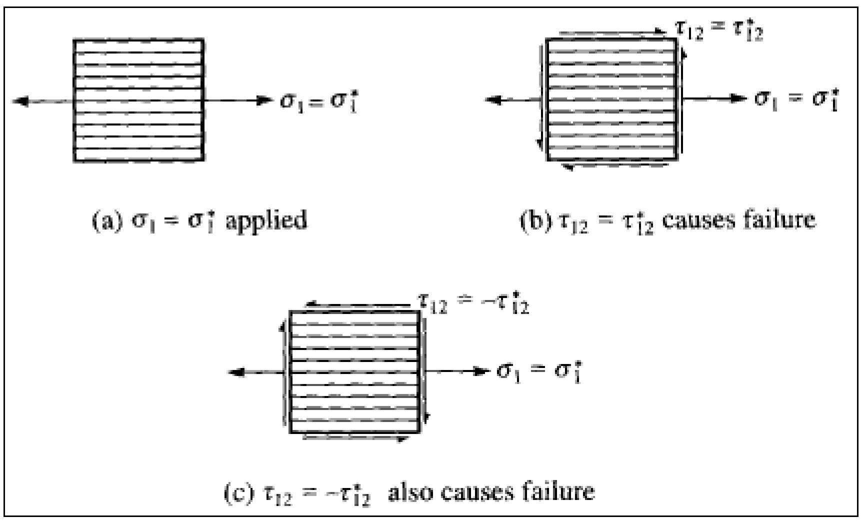


Figure M7.3.4.7 Combined stresses σ_1 and τ_{12} as they apply to the Tsai-Wu criterion

The Tsai-Wu failure criterion thus simplifies one step further to become

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2 = 1 \qquad (7.3.4.31)$$

We now turn to the evaluation of F_{12}.

The coefficient F_{12} involves both σ_1 and σ_2, and to determine F_{12} experimentally requires testing with nonzero values of both σ_1 and σ_1. There are several ways to accomplish this. One is to construct a loading device that applies both σ_1 and σ_2 simultaneously. The resulting biaxial state of stress would be that shown in Figure M7.3.4.8. Ideally the two components of stress would be controlled independently. Theoretically, a single pair of values of σ_1 and σ_2 is all that is needed to determine the value of F_{12}. In practice, a range of values of σ_1 and a range of values of σ_2, including both tensile and compressive values should be studied and an average value computed for F_{12}. A second method of determining F_{12} experimentally is to use a uniaxial specimen with its fibers aligned at some known angle relative to the load direction. Then, as in Figure M7.3.4.9, the uniaxial stress is σ_x, and the components of stress in the principal material system, σ_1, σ_2 and τ_{12}, are given by the transformation equations, equation (7.3.4.31(a)).

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad (7.3.4.31(a))$$

If for a given off-axis angle $\theta = \theta^*$, failure occurs when $\sigma_x = \sigma_x^*$, then for this condition the Tsai-Wu failure criterion of equation (7.3.4.31) becomes

$$\left(F_1 \cos^2(\theta^*) + F_2 \sin^2(\theta^*)\right) \sigma_x^* + \left(F_{11} \cos^4(\theta^*) + F_{22} \sin^4(\theta^*) + F_{66} \cos^2(\theta^*) \sin^2(\theta^*) + 2F_{12} \cos^2(\theta^*) \sin^2(\theta^*)\right) \sigma_x^{*2} = 1 \qquad (7.3.4.32)$$

From this, because the values of all quantities in the equation except F_{12} are known, the value of F_{12} can be determined. Obviously all that is needed is one test. However, several tests should be conducted at different angles to determine the consistency of the value of F_{12} determined in this manner. Finally, F_{12} could be determined by using a helically wound cylinder made from the material of interest. As Figure M7.3.4.10 shows, a cylinder can be internally or externally pressurized, loaded axially, loaded in torsion, or stressed in any combination of these to produce a variety of magnitudes and signs of σ_1, σ_2, and τ_{12} in the cylinder wall. The value of F_{12} can then is studied with such a specimen.

Another method of determining the value of F_{12} appeals to heuristic arguments and is as follows: For the case of plane stress, the von Mises criterion, equation (7.3.4.1), can be written

$$\left(\frac{1}{\sigma^Y}\right)^2 \sigma_1^2 + \left(\frac{1}{\sigma^Y}\right)^2 \sigma_2^2 - \left(\frac{1}{\sigma^Y}\right)^2 \sigma_1 \sigma_2 = 1 \qquad (7.3.4.33)$$

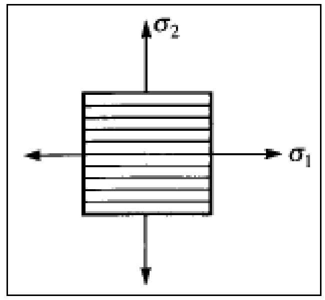


Figure M7.3.4.8 Biaxial loading for determining F_{12} in the Tsai-Wu criterion

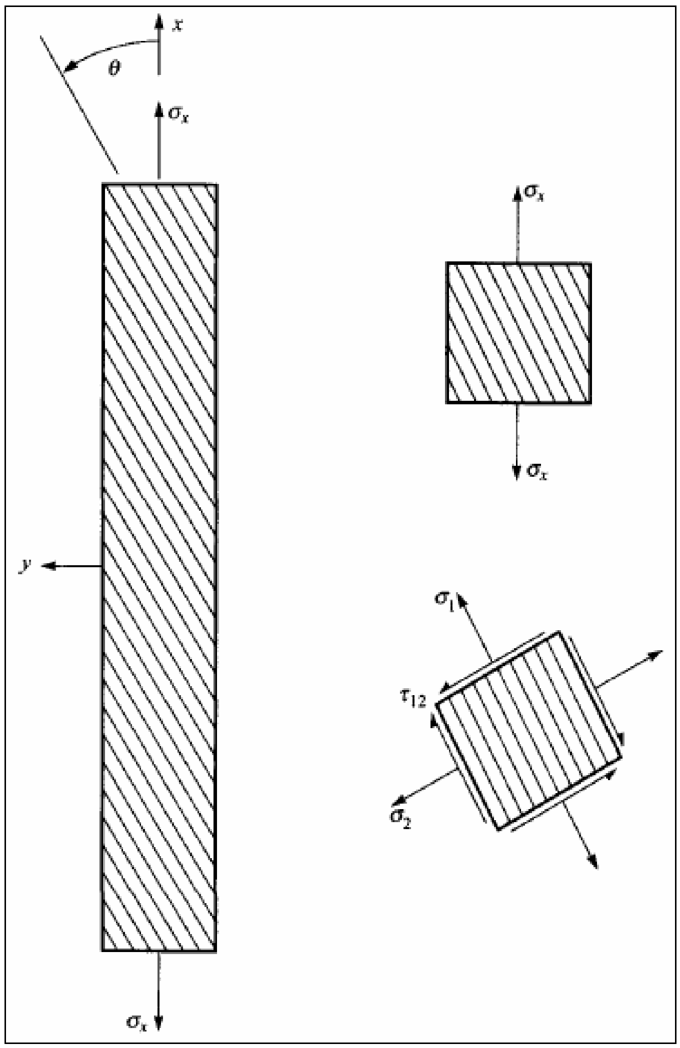


Figure M7.3.4.9 Off-axis tensile specimen for determining F_{12} in the Tsai-Wu criterion

If the Tsai-Wu criterion is applied to cases for which the von Mises criterion is valid, and the subscripts 1 and 2 in the Tsai-Wu criterion are identified with principal stress directions, then

$$\sigma_1^T = \sigma^Y \qquad \sigma_1^C = -\sigma^Y$$
$$\sigma_2^T = \sigma^Y \qquad \sigma_2^C = -\sigma^Y \qquad (7.3.4.34)$$

By the definitions in equations (7.3.4.14) and (7.3.4.19), F_1 and F_2 are results as zero. Because τ_{12} is zero when 1 and 2 are identified with principal stress directions, the Tsai-Wu criterion as given by equation (7.3.4.31) reduces to

$$\left(\frac{1}{\sigma^Y}\right)^2 \sigma_1^2 + \left(\frac{1}{\sigma^Y}\right)^2 \sigma_2^2 + 2F_{12}\sigma_1\sigma_2 = 1 \qquad (7.3.4.35)$$

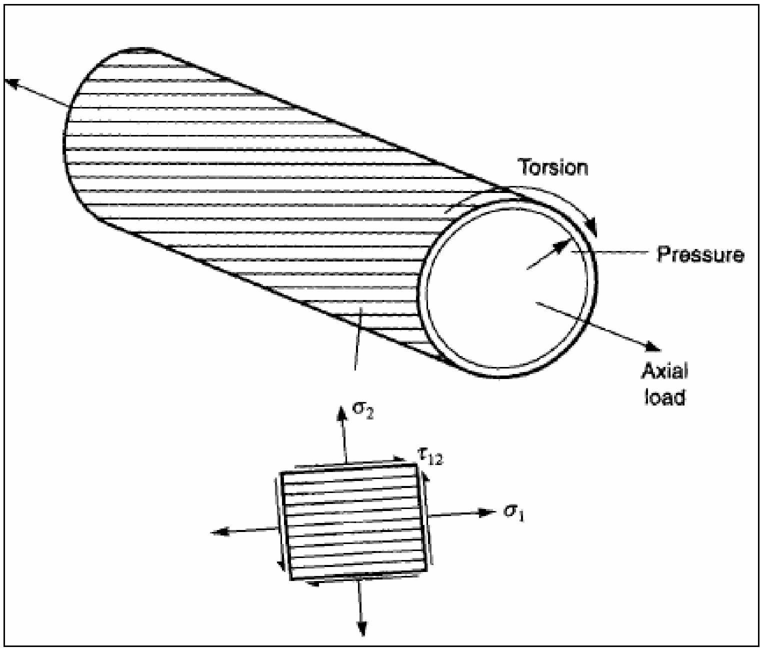


Figure M7.3.4.10 Cylindrical specimen for determining F_{12} in the Tsai-Wu criterion

For this reduced form of the Tsai-Wu criterion to yield the same result as the von Mises criterion, equation (7.3.4.33), it must be that

$$2F_{12} = -\left(\frac{1}{\sigma^Y}\right)^2 \qquad (7.3.4.36)$$

This will be the case if in the Tsai-Wu criterion F_{12} is given by (see Suggested Readings)

$$F_{12} = -\frac{1}{2}\sqrt{F_{11}F_{22}} \qquad (7.3.4.37)$$

Using this relation for composite materials, the Tsai-Wu criterion becomes

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 - \sqrt{F_{11}F_{22}}\sigma_1\sigma_2 = 1 \qquad (7.3.4.38)$$

where,

$$F_1 = \left(\frac{1}{\sigma_1^T} + \frac{1}{\sigma_1^C}\right) \qquad F_{11} = -\frac{1}{\sigma_1^T\sigma_1^C}$$

$$F_2 = \left(\frac{1}{\sigma_2^T} + \frac{1}{\sigma_2^C}\right) \qquad F_{22} = -\frac{1}{\sigma_2^T\sigma_2^C}$$

$$F_{66} = \left(\frac{1}{\tau_{12}^F}\right)^2 \qquad (7.3.4.39)$$

This is the form of the Tsai-Wu criterion we shall use. Table M7.3.4.1 gives the values of the failure stresses and the values of $F_1,\ldots\ldots,F_{66}$ for the graphite-reinforced material used throughout. Note the units of $F_1,\ldots\ldots,F_{66}$.

σ_1^T = 1500 MPa	F_1 = 0.1333 1/GPa
σ_1^C = −1250 MPa	F_2 = 15.00 1/GPa
σ_2^T = 50 MPa	F_{11} = 0.533 (1/GPa)^2
σ_2^C = −200 MPa	F_{22} = 100 (1/GPa)^2
τ_{12}^F = 100 MPa	F_{66} = 100 (1/GPa)^2

Table M7.3.4.1 Tsai-Wu failure parameters for graphite-reinforced composite

In the space formed by $\sigma_1 - \sigma_2 - \tau_{12}$ the Tsai-Wu criterion is an ellipsoid (see Figure M7.3.4.11). The ellipsoid is very long and slender, indicating the strong dependence on direction of the high strength of the fibers, and the weak strength of the matrix material. For the case of no shear in the principal material system, $\tau_{12} = 0$ and the Tsai-Wu criterion is an ellipse in $\sigma_1 - \sigma_2$ space; Figure M7.3.4.12 shows the ellipse for the graphite-reinforced material considered here. The intersections of the ellipse with the coordinate axes are indicated by the letters A, B, C, and D, where these points represent the basic failure stresses $\sigma_1^T, \sigma_2^T, \sigma_1^C$, and σ_2^C, respectively. In the third quadrant the criterion predicts that compressive failure stresses in the fiber direction much more negative than σ_1^C are possible in the presence of compression in the 2 direction. This is a clear example of stress interaction; in this case the interaction predicts a significant strengthening influence. This characteristic of many interactive failure criteria causes concern.

Figure M7.3.4.13 shows several cross sections of the Tsai-Wu ellipsoid with the σ_2 axis expanded. The $\tau_{12} = 0$ case is a reproduction of Figure M7.3.4.12. The stress interaction is evident in all quadrants; the third quadrant again indicates that compressive stresses in the fiber direction exceeding σ_1^C in magnitude are possible with compressive σ_2. The other three quadrants predict that stress interaction effects can degrade strength. For example, with $\tau_{12} = 0$ and $\sigma_2 = -150\,\text{MPa}$, failure in the fiber direction is predicted at just over $\sigma_1 = +700\,\text{MPa}$. With no σ_2, failure in the fiber direction is predicted to occur, of course, at 1500 MPa. The cross sections for nonzero values of τ_{12} indicate that the presence of shear stress τ_{12} even further degrades the levels of σ_1 and σ_2 that can be tolerated.

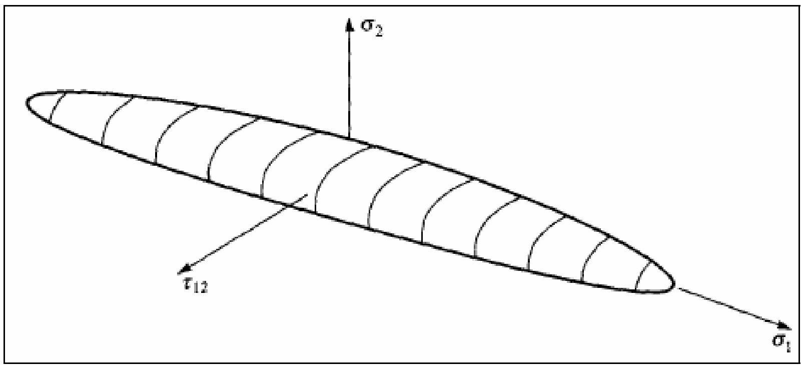


Figure M7.3.4.11 Tsai-Wu ellipsoid in principal material system stress space

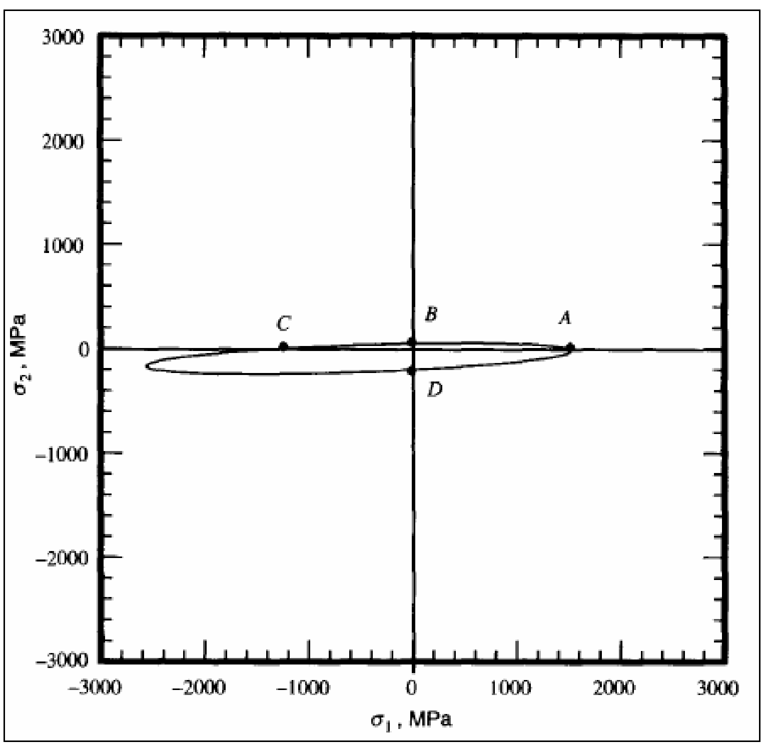


Figure M7.3.4.12 Cross section in $\sigma_1 - \sigma_2$ plane $(\tau_{12} = 0)$ of the Tsai-Wu ellipsoid for the graphite-reinforced composite

Before we turn to example problems to demonstrate the utility of the Tsai-Wu criterion, we should note three important characteristics of the criterion. First, in contrast to the six equations required to apply the maximum stress criterion, equation (7.3.1.4), the Tsai-Wu criterion involves only one equation, equation (7.3.4.38). This makes the application of the Tsai-Wu criterion simpler than the application of the maximum stress criterion. Second, because the Tsai-Wu criterion involves powers and products of the stresses, whenever the criterion is used to compute a failure load, the criterion will yield two answers, one positive and one negative. This will be apparent in the example problems to follow. However, this characteristic can be demonstrated with a rather simple example that, to some degree, represents a trivial application of the criterion: Consider a single layer of material subjected to a stress a in the fiber direction. We ask what value of a cause's failure. For this case,

$$\sigma_1 = \sigma \qquad \sigma_2 = 0 \qquad \tau_{12} = 0 \tag{7.3.4.40}$$

and the Tsai-Wu criterion becomes

$$F_1\sigma + F_{11}\sigma^2 = 1 \tag{7.3.4.41}$$

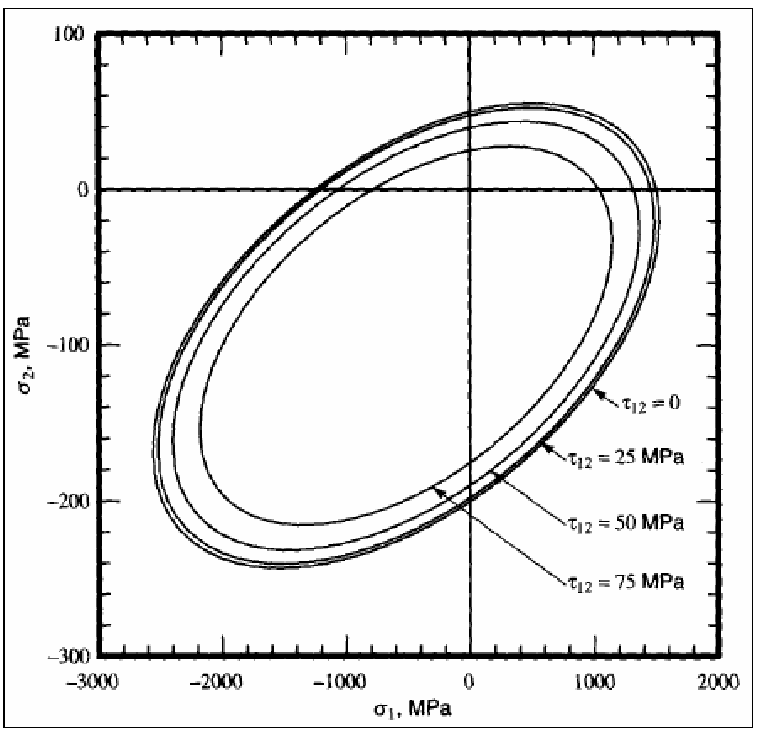


Figure M7.3.4.13 Several cross sections of the Tsai-Wu ellipsoid

This can be rearranged to read

$$F_{11}\sigma^2 + F_1\sigma - 1 = 0 \quad (7.3.4.42)$$

This is a quadratic equation for a and solution leads to

$$\sigma = \frac{-F_1 \pm \sqrt{F_1^2 + 4F_{11}}}{2F_{11}} \quad (7.3.4.43)$$

Substituting for F_1 and F_{11} leads to

$$\sigma = \sigma_1^T \quad \text{and} \quad \sigma = \sigma_1^C \quad (7.3.4.44)$$

In this case the stress to cause failure in tension and the stress to cause failure in compression, in the fiber direction, are the answers. This was expected for this problem but the results indicate the characteristic of the criterion to yield two answers. For this reason, the Tsai-Wu criterion is in a class of criteria called quadratic failure criteria. Finally, unlike the maximum stress criterion, the Tsai-Wu criterion does not directly indicate the mode of failure. The criterion predicts failure but does not indicate whether failure is due to fiber failure, shear failure, and so on. With additional calculations, however, some indication of the mode of failure is possible.

Exercises for Section 7.3.4.1

1. Verify the numerical values of the failure parameters in Table M7.3.4.1 and construct a similar table for glass-reinforced composite from the data in Table M7.3.1.1.
2. What would be the differences in Figure M7.3.4.13 if the ellipses from the Tsai-Wu criterion are drawn for the case of t\i = —25 MPa, —50 MPa, and —75 MPa?

7.3.4.2 Failure Example: Tube with Axial Load — Tsai-Wu Criterion

As with Failure Example 1, consider again a tube with a mean radius of 25 mm made of graphite-reinforced material that has a 10-layer wall with a stacking sequence of $[\mp 20/0_3]_S$.
The tube is designed to resist axial load but has the low-angle off-axis layers (the ±20° layers) to provide some circumferential and torsional stiffness, and to hold the load-carrying layers together. If we use the Tsai-Wu failure criterion, what is the maximum allowable axial load? What layer or layers control failure? What is the mode of failure? How do the predictions compare with those of the maximum stress criterion?
The tube and loading were illustrated in Figure M7.3.1.8, which was used in connection with studying this problem in the context of the maximum stress failure criterion. As with the procedure when using the maximum stress criterion, the stresses in the principal material system in each layer are computed for the case of a unit load $P = 1\text{N}$. The load is assumed to be multiplied by p, and so the stresses in each layer are also multiplied by p. The primary question

is to determine the value or values of p that, according to the Tsai-Wu criterion, cause the tube to fail. These can be determined by substituting the stresses due to load p into the Tsai-Wu criterion and determining the value of p that causes the criterion to equal unity. Of course the stresses in each layer are different and so there is an equation, involving p, for each layer. The equation that leads to the lowest value of p indicates which layer controls failure, and the value of p from this equation is the failure load.

Table M7.3.1.2 presented the stresses in each layer due to a unit axial load, and Table M7.3.1.3 gave the stresses due to a load p. For the +20° layers, then, the Tsai-Wu criterion, equation (7.3.4.38), predicts failure to occur when

$$F_1(3830p) + F_2(-112.3p) + F_{11}(3830p)^2 + F_{22}(-112.3p)^2 + F_{66}(-148.7p)^2 - \sqrt{F_{11}F_{22}}(3830p)(-112.3p) = 1 \qquad (7.3.4.45)$$

which leads to a quadratic equation of the form

$$Ap^2 + Dp + F = 0 \qquad (7.3.4.46)$$

Using the values of the coefficients $F_1, \ldots\ldots, F_{66}$ from Table M7.3.4.1 results in

$$A = 0.1444 \times 10^{-10}$$
$$D = -22\,000 \times 10^{-10}$$
$$F = -1 \qquad (7.3.4.47)$$

and solving for p leads to

$$p = -198\,000 \quad \text{and} \quad p = +350\,000 \qquad (7.3.4.48)$$

Thus the Tsai-Wu criterion predicts that the 20° layers will fail due either to a tensile load of+350 kN or to a compressive load of —198 kN. The simplicity of the Tsai-Wu approach can certainly be appreciated. One equation, equation (7.3.4.46), as compared to six, is much easier to deal with.

Using the stresses in the —20° layers due to a load p, from Table M7.3.1.3, the Tsai-Wu criterion predicts failure to occur when

$$F_1(3830p) + F_2(-112.3p) + F_{11}(3830p)^2 + F_{22}(-112.3p)^2 + F_{66}(148.7p)^2 - \sqrt{F_{11}F_{22}}(3830p)(-112.3p) = 1 \quad (7.3.4.49)$$

where this equation is, again, of the form

$$Ap^2 + Dp + F = 0 \quad (7.3.4.50)$$

with

$$A = 0.1444\times10^{-10}$$
$$D = -22000\times10^{-10} \quad (7.3.4.51)$$
$$F = -1$$

This equation is identical with the equation for the +20° layers and the roots are thus

$$p = -198\,000 \quad \text{and} \quad p = +350\,000 \quad (7.3.4.52)$$

The load levels that cause the +20° layers to fail also cause the —20° layers to fail. Note that we have not discussed the mode of failure. Unlike the maximum stress criterion, the Tsai-Wu criterion does not explicitly address failure mode. However, shortly we shall address failure mode from the point of view of the Tsai-Wu criterion. Recall from Table M7.3.1.6 that the maximum stress criterion predicted the +20° layers would fail at a tensile load of $P = +392\,\text{kN}$ due to tensile stresses in the fiber direction and at compressive load of $P = -327\,\text{kN}$ due to compressive stresses in the fiber direction. As Table M7.3.1.6 shows, these same tensile and compressive load levels and failure modes were predicted for the —20° layers by the maximum stress criterion. These levels are larger in absolute value than the +350 kN and —198 kN predicted by the Tsai-Wu criterion. This is due to strength-reducing interactive effects in particular octants in $\sigma_1 - \sigma_2 - \tau_{12}$ space with the Tsai-Wu criterion.

The failure load for the 0° layers is determined in a similar manner. Using the stresses in the 0° layers due to load p in the Tsai-Wu criterion leads to

$$F_1(4770p) + F_2(-168.6p) + F_{11}(4770p)^2 + F_{22}(-168.6p)^2 + F_{66}(0p)^2 - \sqrt{F_{11}F_{22}}(4770p)(-168.6p) = 1 \quad (7.3.4.53)$$

Again a quadratic of the form

$$Ap^2 + Dp + F = 0 \quad (7.3.4.54)$$

results, with

$$A = 0.208 \times 10^{-10}$$
$$D = -31\,600 \times 10^{-10}$$
$$F = -1 \qquad (7.3.4.55)$$

leading to roots

$$p = -156\,000 \quad \text{and} \quad p = +308\,000 \qquad (7.3.4.56)$$

For comparison, the maximum stress failure criterion predicts the values of $P = -262\,\text{kN}$ and $P = +315\,\text{kN}$ for the 0° layers. Table M7.3.4.2 summarizes the values of p for the three layer orientations, and the predictions of the maximum stress criterion are in parenthesis. Again, the primary reason for the difference between the two criteria is the stress interaction effects inherent in the Tsai-Wu criterion and absent from the maximum stress criterion. On an overall basis, the Tsai-Wu criterion predicts a tensile failure load of +308 kN, as opposed to +315 kN for the maximum stress criterion, and a compression failure load of —156 kN, as opposed to —262 kN. While the tensile load predictions are somewhat close, the compressive load predictions are significantly different. Which one is correct? Actually, neither is correct. Neither is incorrect. If, as discussed earlier, failure criteria are viewed as indicators rather than absolute predictors, then having two answers is not so disturbing.

To study the stress interaction effects, and the issue of failure mode as predicted by the Tsai-Wu criterion, consider the following: At the failure load level, the left-hand side of equation (7.3.4.38) sums to unity. Each of the six terms on the left-hand side contributes to, or subtracts from, the trend toward unity, depending on the F_i, F_{ij} values and the values of $\sigma_1 - \sigma_2$, and τ_{12} at failure. The contribution to unity of each of the six terms at the failure load level can easily be computed. For example, using the two values of p computed from equation (7.3.4.45) for the +20° layers, we can construct a table to show the contribution of each term on the left-hand side of equation (7.3.4.38). Table M7.3.4.3 shows the results of these calculations for the two values of p. The values of σ_1, σ_2, and τ_{12} in the table are a result of using the appropriate value of p and the entries of Table M7.3.1.3. For the condition of a tensile failure load, $P = +350\,\text{kN}$, it appears that the term $F_{11}\sigma_1^2$ of contributes significantly to the value of unity. However, the interaction term $-\sqrt{F_{11}F_{22}\sigma_1\sigma_2}$ also contributes, as does the shear term $F_{66}\tau_{12}^2$. The term F2O2 tends to subtract. From these results, therefore, failure in the +20° layers when the applied load is tensile appears to be dominated by tensile failure σ_1, but with some shear effects due to τ_{12} and interaction with compressive σ_2. Turning to Table M7.3.1.6, we can see that, according to the maximum stress criterion, in the +20° layers fiber direction tension controls the level of applied tensile load. According to the Tsai-Wu criterion, this failure mode is of primary importance, but interaction lowers the tensile load level relative to the maximum stress criterion level, that is, +392 kN versus +350 kN.

Layer	p (negative)	p (positive)
+20°	−198 (−327)	+350 (+392)
−20°	−198 (−327)	+350 (+392)
0°	−156 (−262)	+308 (+315)

* Maximum stress criterion prediction in parenthesis.

Table M7.3.4.2 Summary of loads P (kN) to cause failure in $\left[\mp 20/0_3\right]_S$ tube: Tsai-Wu criterion*

	$p = -198\ 000$ N	$p = +350\ 000$ N
σ_1, MPa	−758	1340
σ_2, MPa	22.2	−39.3
τ_{12}, MPa	29.4	−52.1
$F_1\sigma_1$	0.101	−0.179
$F_2\sigma_2$	0.334	−0.590
$F_{11}\sigma_1^2$	0.306	0.958
$F_{22}\sigma_2^2$	0.049	0.155
$F_{66}\tau_{12}^2$	0.087	0.271
$-\sqrt{F_{11}F_{22}}\sigma_1\sigma_2$	0.123	0.385

Table M7.3.4.3 Contribution of terms in Tsai-Wu criterion for +20° layers in $\left[\pm 20/0_3\right]_S$ tube subject to tension

For a compressive applied load, $P = -198\,\text{kN}$, the $F_2\sigma_2$ and $F_{11}\sigma_1^2$ terms dominate, with the $F_1\sigma_1$ and $-\sqrt{F_{11}F_{22}\sigma_1\sigma_2}$ terms contributing. These four terms themselves nearly add to unity, signifying that interaction between these two components is important. The idea that interaction is an issue is further reinforced when the —198 kN level of allowable compressive load for the Tsai-Wu criterion is compared with the —327 kN applied load of the maximum stress criterion. The —327 kN level prediction is based on a fiber direction compression stress failure with no interaction assumed.

Table M7.3.4.4 summarizes the contributions of the terms in the Tsai-Wu failure criterion for each of the three-layer orientations in the $\left[\pm 20/0_{30}\right]_S$ tube at the failure load levels for each layer. For the 0° layers at a tensile load of $P = +308\,\text{kN}$, the largest contribution to the unity value of the Tsai-Wu criterion is due to $F_{11}\sigma_1^2$. However, the interaction term $-\sqrt{F_{11}F_{22}\sigma_1\sigma_2}$ adds, while the $F_1\sigma_1$, $F_2\sigma_2$ terms subtract. It could be stated that failure in the 0° layers due to an applied tensile load is due to fiber failure, with some interaction effects with σ_2. For a compressive axial load, the three largest terms are $F_2\sigma_2, F_{11}\sigma_1^2$, and the interaction term. Because compressive axial loads cause tensile σ_2 in the 0° layers, interaction between σ_1 and σ_2 might well be expected.

In summary, the results of Table M7.3.4.4 indicate the tube is limited in tension to $P = +308\,\text{kN}$ due to tensile failure in the fiber direction in the 0° layers. The tube is limited in compression to $P = -156\,\text{kN}$ due to an interaction between σ_1 and σ_2 in the 0° layers. Though the limiting number from the maximum stress criterion for a tensile value of P is similar (+315 kN), the compression value of p is quite different (—156 kN versus —262 kN). The 0° layers are predicted to be the critical layers for both criteria.

	+20° layers		–20° layers		0° layers	
	$P = -198$	$P = +350$	$P = -198$	$P = +350$	$P = -156.0$	$P = +308$
σ_1, MPa	−758	1340	−758	1340	−743	1467
σ_2, MPa	22.2	−39.3	22.2	−39.3	26.3	−51.9
τ_{12}, MPa	29.4	−52.1	29.4	−52.1	0	0
$F_1\sigma_1$	0.101	−0.179	0.101	−0.179	0.099	−0.196
$F_2\sigma_2$	0.334	−0.590	0.334	−0.590	0.394	−0.779
$F_{11}\sigma_1^2$	0.306	0.958	0.306	0.958	0.295	1.149
$F_{22}\sigma_2^2$	0.049	0.155	0.049	0.155	0.069	0.269
$F_{66}\tau_{12}^2$	0.087	0.271	0.087	0.271	0	0
$-\sqrt{F_{11}F_{22}}\sigma_1\sigma_2$	0.123	0.385	0.123	0.385	0.143	0.556
Total	1.000	1.000	1.000	1.000	1.000	1.000

The maximum stress failure criterion predicts failure with a positive axial load of $P = +315\,\text{kN}$; failure is due to tensile stresses in the fiber direction in the 0° layers. Failure with a negative axial load occurs when $P = -262\,\text{kN}$ and is due to compression stresses in the fiber direction in the 0° layers.

Table M7.3.4.4 Summary of loads P (kN) to cause failure in $\left[\pm 20/0_3\right]$ tube: Tsai-Wu criterion*

It should be mentioned that using the contributions in the Tsai-Wu criterion to indicate the failure mode is not as definitive as the failure mode prediction of the maximum stress criterion. However, using the contributions of the Tsai-Wu criterion to indicate possible failure modes is in keeping with the spirit of using failure criteria as indicators of failure rather than as absolute predictors.

Before continuing to the second example, it is logical to ask the significance of the negative contributions to the Tsai-Wu value of unity—that is, $F_1\sigma_1 = -0.196$ for the 0° layers with $P = +308\,\text{kN}$. Figure M7.3.4.12 showed enhanced compressive strength in me fiber direction due to the presence of a compressive σ_2—that is, the elongation of the ellipse beyond $\sigma_1 = \sigma_1^C$ in the third quadrant. When such strengthening interaction is predicted, it appears as a subtraction from the sum toward unity of the terms on the left-hand side of the Tsai-Wu criterion.

Exercises for Section 7.3.4.2

1. Suppose the off-axis layers in the tube of Failure Example 1 were at ±30° instead of ±20°.
   a. Based on the Tsai-Wu criterion, what would be the axial load capacity of the tube?
   b. Would the failure mode and the layers that control failure be the same as when the fibers were at ±20°? (To answer this question you essentially must redo the example problem, starting with the stresses due to P = 1 N, i.e., Table M7.3.1.2.)
   c. Compare the results with the predictions of the maximum stress criterion, Exercise 1 in the Exercises for Section 7.3.1.2.
2. A $\left[\pm 45/0_2\right]_S$ graphite-reinforced plate is subjected to a biaxial loading such that the stress resultant in the y direction is opposite in sign to and one-half the magnitude of the stress resultant in the x direction. Call the stresses resultant in the x direction N; the loading is given by

$$N_x = N$$
$$N_y = -0.5N$$
$$N_{xy} = 0$$

(a) Use the Tsai-Wu criterion to compute the value of N to cause failure,
(b) What layers control failure?
(c) Use the contribution to unity of each term in the criterion to estimate the failure mode. To answer these questions, construct a table similar to Table M7.3.4.4.
(d) Compare the results with the predictions of the maximum stress criterion, Exercise 2 in the Exercises for Section 7.3.1.2.

M7.3.4.3 Failure Examples 8: Tube in Torsion —Tsai-Wu Criterion

Turn now to the second example solved previously with the maximum stress criterion, and consider that the $\left[\pm 20/0_3\right]_S$ tube is subjected to 225 kN tension and that there is an unwanted amount of torsion, T. According to the Tsai-Wu failure criterion, what is the maximum amount of torsion the tube can withstand before it fails? What layer or layers control failure and what is the mode of failure? How do the predictions compare with those of the maximum stress criterion?

The situation was illustrated in Figure M7.3.1.9, and as we were when considering the maximum stress failure criterion, we shall be interested in the stresses in each layer due to the 225 kN axial load, and an unknown to be solved for the amount of torsion t . Table M7.3.1.9 provided us with the stresses in each layer for this situation. Referring to this table, then, for the +20° layers, we find that the Tsai-Wu criterion, equation (7.3.4.38), becomes

$$\begin{aligned} F_1(861 + 0.804t) \times 10^6 &+ F_2(-25.3 - 0.048t) \times 10^6 \\ &+ F_{11}(861 + 0.804t)^2 \times 10^{12} \\ &+ F_{22}(-25.3 - 0.048t)^2 \times 10^{12} \\ &- \sqrt{F_{11}F_{22}}(861 + 0.804t)(-25.3 - 0.048t) \times 10^{12} \\ &+ F_{66}(-33.5 + 0.0552t)^2 \times 10^{12} = 1 \end{aligned} \tag{7.3.4.57}$$

Substituting for the values of $F_1, ..., F_{66}$ results in a quadratic equation of the form

$$Bt^2 + Et + F = 0 \tag{7.3.4.58}$$

Here F , rather than being —1, as in equation (7.3.4.46), involves the stresses due to the applied axial load, F_i and F_{ij} . In this case

$$\begin{aligned} B &= 1.163 \times 10^{-6} \\ E &= 235 \times 10^{-6} \\ F &= -0.763 \end{aligned} \tag{7.3.4.59}$$

and the solution of equation (7.3.4.58) results in

$$t = -917 \quad \text{and} \quad t = +715 \tag{7.3.4.60}$$

For the —20° layers, the Tsai-Wu criterion becomes

$$\begin{aligned} F_1(861 - 0.804t) \times 10^6 &+ F_2(-25.3 + 0.048t) \times 10^6 \\ &+ F_{11}(861 - 0.804t)^2 \times 10^{12} \\ &+ F_{22}(-25.3 + 0.048t)^2 \times 10^{12} \\ &- \sqrt{F_{11}F_{22}}(861 - 0.804t)(-25.3 + 0.0408t) \times 10^{12} \\ &+ F_{66}(33.5 + 0.0552t)^2 \times 10^6 = 1 \end{aligned} \tag{7.3.4.61}$$

resulting in

$$t = -715 \quad \text{and} \quad t = +917 \tag{7.3.4.62}$$

Finally, following the above procedure for the 0° layers leads to

$$t = -1125 \quad \text{and} \quad t = +1125 \tag{7.3.4.63}$$

The results of this analysis are summarized in Table M7.3.4.5, and the contributions to unity for each layer and the six torsional load levels are included in the table.

In Table M7.3.4.5 we see that a torsional load of $T = \pm 715\,\text{Nm}$ causes failure in the ±20° layers due to fiber tension. The stress component σ_1 is near its failure level of 1500 MPa, and the $F_{11}\sigma_1^2$ term is near unity. This failure load level is close to the prediction of ±795 Nm from the maximum stress criterion, Table M7.3.1.11, and the predicted failure mode is the same.

	+20° layers		−20° layers		0° layers	
	$T = -917$	$T = +715$	$T = -715$	$T = +917$	$T = -1125$	$T = +1125$
σ_1, MPa	123.7	1436	1436	123.7	1072	1072
σ_2, MPa	18.86	59.7	−59.7	18.86	−37.9	−37.9
τ_{12}, MPa	−84.1	6.06	−6.06	84.1	−81.1	81.1
$F_1\sigma_1$	−0.016	−0.192	−0.192	−0.016	−0.143	−0.143
$F_2\sigma_2$	0.283	−0.896	−0.896	0.283	−0.569	−0.569
$F_{11}\sigma_1^2$	0.008	1.100	1.100	0.008	0.613	0.613
$F_{22}\sigma_2^2$	0.036	0.357	0.357	0.036	0.144	0.144
$F_{66}\tau_{12}^2$	0.707	0.004	0.004	0.707	0.658	0.658
$-\sqrt{F_{11}F_{22}}\sigma_1\sigma_2$	−0.017	0.627	0.626	−0.017	0.297	0.297
Total	1.000	1.000	1.000	1.000	1.000	1.000

*The maximum stress criterion predicts failure for a positive and negative torsion of $T = \pm 795$ N·m due to tensile failure in the fiber direction in the ±20° layers.

Table M7.3.4.5 Summary of torsions T (N-m) to cause failure in [±20/03] tube with $P = 0.225\,\text{MN}$: Tsai-Wu criterion*

The maximum stress criterion predicts failure for a positive and negative torsion of T = +795 Nm due to tensile failure in the fiber direction in the +20° layers.

Exercise for Section 7.3.4.3

A $\left[\pm 45/0_2\right]_S$ graphite reinforced plate is subjected to a biaxial loading such that the stress resultant in the y direction is —0.200 MN/m and the stress resultant in the x direction is variable, that is:

$$N_x = N$$
$$N_y = -0.200 MN/m$$
$$N_{xy} = 0$$

(a) Use the Tsai-Wu failure criterion to compute the value of N required to cause failure.
(b) What layer or layers control failure?
(c) What is the estimated mode of failure?
(d) Compare your results with the predictions of the maximum stress criterion in the Exercise for Section 7.3.1.3.

M7.3.4.4 Failure Example 9: Tube with Combined Load —Tsai-Wu Criterion

The third example demonstrates an advantage of the single-equation viewpoint of the Tsai-Wu failure criterion. The maximum stress criterion involves six equations per layer and, when interpreted graphically, leads to somewhat complicated figures (e.g., Figure M7.3.1.18). Consider the third example with the Tsai-Wu failure criterion applied rather than the maximum stress criterion. Specifically, in a particular application the tube is being used with an axial load P and a small torsional load T. According to the Tsai-Wu criterion, what are the ranges of applied axial load and applied torque the tube can withstand before it fails? What layer or layers control failure? What is the mode of failure? How do the results compare with those of the maximum stress criterion?

As stated at the time we studied this problem in the context of the maximum stress criterion, this is truly a combined-stress problem. The stresses in each layer due to an applied axial load p and a torsion t were given in Table M7.3.1.12. In this case, both p and t are unknown and we wish to establish a relationship between them based on the Tsai-Wu criterion. There will be a relation for each layer, and in p-t space this relation will describe an ellipse. Any point on the ellipse represents combinations of p and t that will produce failure in that layer, while any point inside the ellipse represents combinations of p and t that will not cause failure. Since each layer produces an ellipse, in $\mathrm{p-t}$ space there will be three ellipses: one ellipse represents the failure characteristics for the +20° layers, the second ellipse represents the failure characteristics for the —20° layers, and the third ellipse represents the failure characteristics for the 0° layers. The region interior to all three ellipses will represent the combinations of p and t that are "safe" for the tube as a whole. The calculations for these ellipses are as follows:

For the +20° layers, the Tsai-Wu criterion becomes

$$\begin{aligned} &F_1(0.00383p+0.804t)+F_2(-0.0001123p-0.0481t)+F_{11}(0.00383p+0.804t)^2 \\ &+ F_{22}(-0.0001123p - 0.0481t)^2 - \sqrt{F_{11}F_{22}}(0.00383p + 0.804t) \\ &\times (-0.0001123p - 0.0481t) + F_{66}(-0.0001487p + 0.0552t)^2 = 1 \end{aligned} \quad (7.3.4.64)$$

where substituting for the F_i and F_{ij} leads to an equation of the form

$$Ap^2 + Bt^2 + Cpt + Dp + Et + F = 0 \quad (7.3.4.65)$$

In the above,

$$\begin{aligned} A &= 0.1444 \times 10^{-10} \\ B &= 11\,630 \times 10^{-10} \\ C &= 47.3 \times 10^{-10} \\ D &= -22\,000 \times 10^{-10} \\ E &= -82.9 \times 10^{-5} \\ F &= -1 \end{aligned} \quad (7.3.4.66)$$

and in p-t space equation (7.3.4.65) represents an ellipse. For the case of t — 0, this equation reduces to the form of equation (10.46), and to the form of equation (7.3.4.58) when $P = +225\,\text{kN}$. Figure M7.3.4.14 illustrates the ellipse represented by equation (7.3.4.65), and the figure is drawn on the same scale as Figure M7.3.1.15. Figure M7.3.4.14 is the Tsai-Wu criterion counterpart to Figure M7.3.1.15. The skewing of the shaded region toward the fourth quadrant is evident in Figure M7.3.4.14, as it was in Figure M7.3.1.15. The values of p and t which represent load combinations that do not cause failure in the +20° layers are given by all points inside the ellipse. Points on the ellipse represent values of p and t that result in failure of the +20° layers.

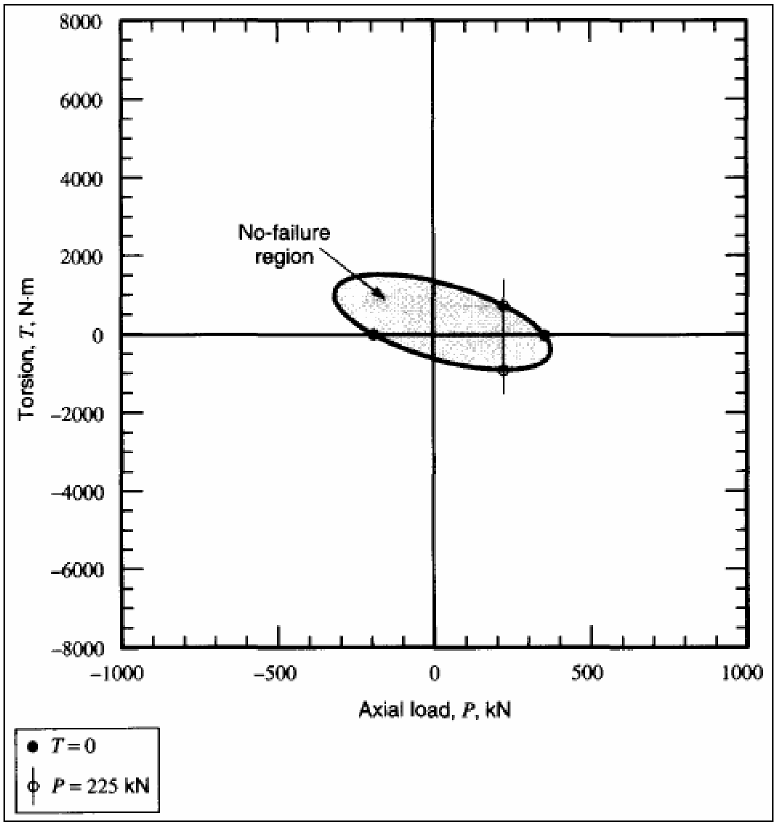


Figure M7.3.4.1 Tsai-Wu failure ellipse for +20° layers

On Figure M7.3.4.14 the points for $T = 0$ are noted. This corresponds to the first example problem and leads to the values of $P = -198.0\,\text{kN}$ and $P = +350\,\text{kN}$. (See equation [7.3.4.48] and the discussion leading up to it.) The points for P = 225 kN, which correspond to the second example, are also shown, resulting in $T = -917\,\text{Nm}$ and $T = +715\,\text{Nm}$. (See equation [7.3.4.60] and the discussion leading up to it.) Of course, for the case of torsion only (P = 0), the intercepts of the ellipse with the vertical axis provide failure information.

Applying the Tsai-Wu criterion to the —20° and 0° layers results in, respectively,

$$\begin{aligned} F_1(0.00383p - 0.804t) &+ F_2(-0.0001123p + 0.0481t) \\ &+ F_{11}(0.00383p - 0.804t)^2 \\ &+ F_{22}(-0.0001123p + 0.0481t)^2 \\ &- \sqrt{F_{11}F_{22}}(0.00383p - 0.804t) \\ &\times (-0.0001123p + 0.0481t) \\ &+ F_{66}(0.0001487p + 0.0552t)^2 = 1 \qquad (7.3.4.67) \end{aligned}$$

$$F_1(0.00477p) + F_2(-0.0001686p) + F_{11}(0.00477p)^2 + F_{22}(-0.0001686p)^2 - \sqrt{F_{11}F_{22}}(0.00477p)(-0.0001686p) + F_{66}(0.0721t)^2 = 1 \quad (7.3.4.68)$$

These are both of the form of equation (7.3.4.65). For the —20° layers

$$\begin{aligned} A &= 0.1444 \times 10^{-10} \\ B &= 11\,630 \times 10^{-10} \\ C &= -47.3 \times 10^{-10} \\ D &= -22\,000 \times 10^{-10} \\ E &= 82.9 \times 10^{-5} \\ F &= -1 \end{aligned} \quad (7.3.4.69)$$

while for the 0° layers

$$\begin{aligned} A &= 0.208 \times 10^{-10} \\ B &= 5200 \times 10^{-10} \\ C &= 0 \\ D &= -31\,600 \times 10^{-10} \\ E &= 0 \\ F &= -1 \end{aligned} \quad (7.3.4.70)$$

Failure Example 9: Tube with Combined Load — Tsai-Wu Criterion

The ellipse represented by equation (7.3.4.67) (the —20° layers) is shown in Figure M7.3.4.15, the counterpart to Figure M7.3.1.16 for the maximum stress criterion, and the ellipse represented by equation (7.3.4.68) (the 0° layers) is shown in Figure M7.3.4.16, the counterpart to Figure M7.3.1.17. Note the interaction predicted by the Tsai-Wu criterion in all four quadrants for these three ellipses, interaction being identified by the smooth, as opposed to sharp, corners of the relations. In Figure 10.17 the ellipse for the +20° layers, the —20° layers, and the 0° layers are superimposed. This figure is the counterpart to Figure 9.18, and the area common to all three ellipses represents values of P and T that will not cause failure in the tube. The common area is bounded on the upper side by the ellipse for the —20° layers and on the lower side by the ellipse for the +20° layers. Except for low values of T combined with extreme values of P, failure in the tube is governed by the ±20° layers; this is somewhat similar to the conclusions reached by studying Figure M7.3.1.18. It is possible to associate various portions of the elliptical boundaries of the shaded region with various modes of failure, or the interaction of various modes, for the various layers. Alternatively, for specific values of P and T, a table similar to Table M7.3.4.4 or Table M7.3.4.5 can be constructed to determine the contributions of $F_1\sigma_1, F_{11}\sigma_1^2$, and the like. In either case, figures such as Figure M7.3.4.17, which show how the various layers contribute to the overall failure envelope, are very useful.

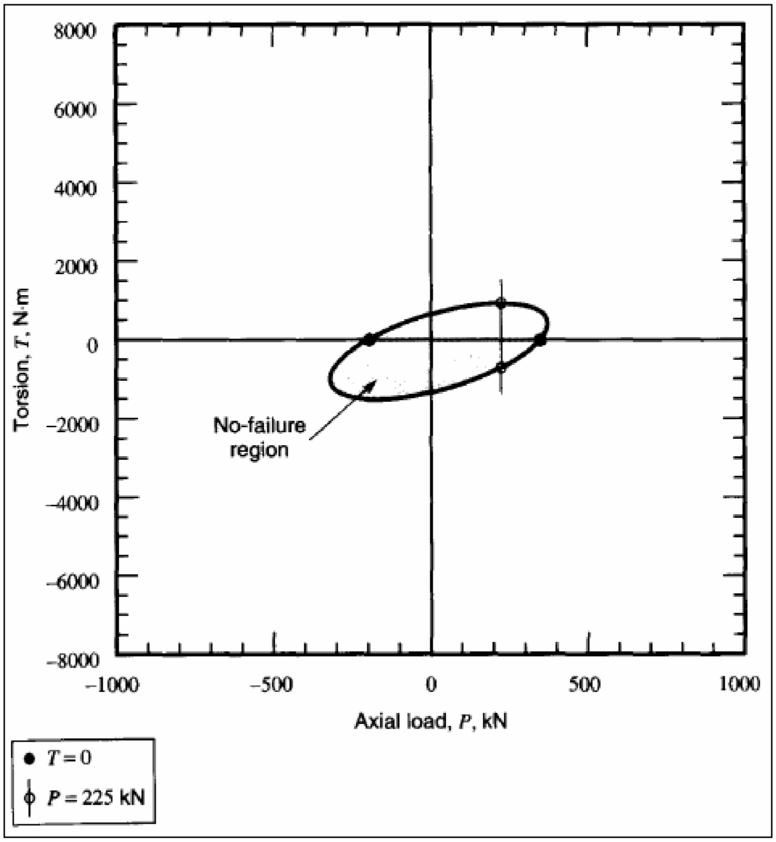


Figure M7.3.4.15 Tsai-Wu failure ellipse for —20° layers

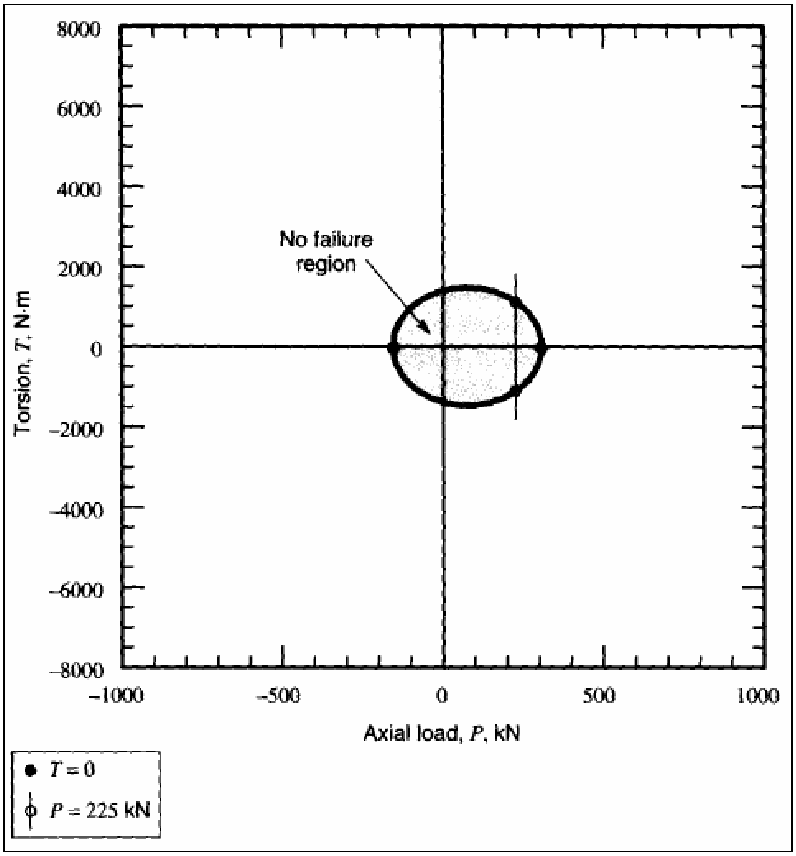


Figure M7.3.4.16 Tsai-Wu failure ellipse for 0° layers

Before we leave the third example, it is interesting to compare the calculations from the Tsai-Wu criterion with those from the maximum stress criterion. Such a comparison is shown in Figure M7.3.4.18 for the 20° layers, and in Figure M7.3.4.19 for the 0° layers. In these figures the two criteria are superposed and the safe regions are indicated. For both layers the Tsai-Wu criterion is mostly within the maximum stress criterion. As stated earlier, the rounding off of the sharp corners, or cusps, of the maximum stress criterion by the Tsai-Wu criterion is due to interaction of the various stress components. The extensions of the Tsai-Wu criterion beyond the maximum stress criterion are also due to interaction effects, for example, negative σ_2 strengthening negative σ_1 in the third quadrant of Figure M7.3.4.12.

Finally, Figure M7.3.4.20 combines the important portions of the maximum stress criterion, Figure M7.3.1.18, and the three ellipses from the Tsai-Wu criterion, Figure M7.3.4.17. The figure is rather complicated but it is clear that for this problem, the region within the three ellipses is within the region defined by the maximum stress criterion. This, as seen from the figures of comparisons for specific layers, such as Figure M7.3.4.18, may not always be the case.

Failure Example 9: Tube with Combined Load — Tsai-Wu Criterion

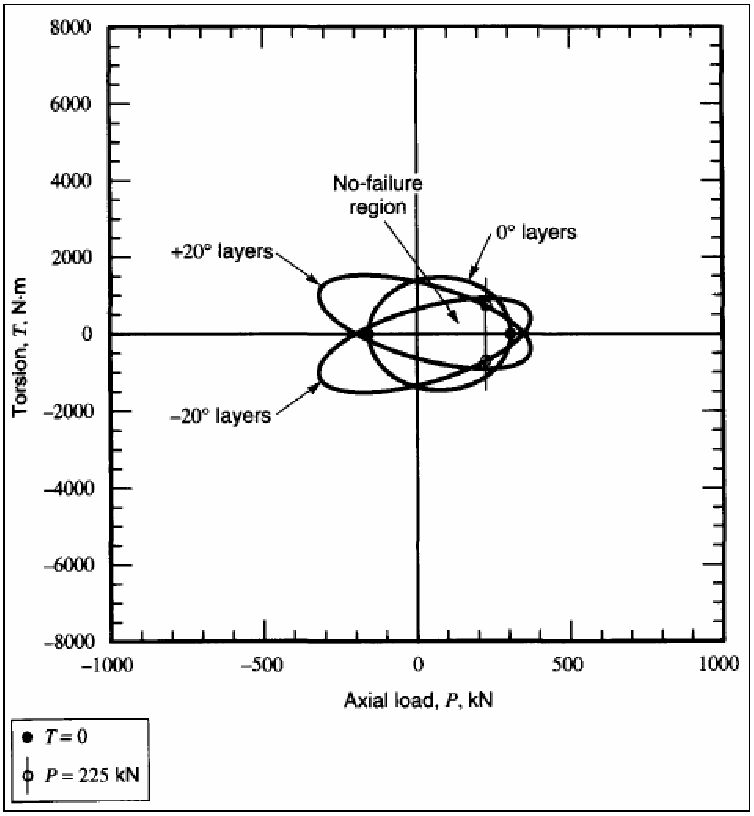


Figure M7.3.4.17 Superposition of the Tsai-Wu failure ellipses for +20°, -20°, and 0° layers

This completes our treatment of the three tube problems, which were studied using two different failure criteria. We shall now turn to solving the three familiar problems involving flat laminates, namely, $[0/90]_S$ and $[\pm 30/0]_S$ laminate subjected to force and moment resultants.

Exercise for Section 7.3.4.4

A $[\pm 45/0_2]_S$ graphite-reinforced plate is subjected to a biaxial loading; that is:

$$N_x = N_x$$
$$N_y = N_y$$
$$N_{xy} = 0$$

a. Use the Tsai-Wu failure criterion to determine the no-failure region of the plate in $N_x - N_y$ space. To do this, plot the ellipses for the various layers and indicate the safe region within the ellipses. Note that the cases of Exercise 2 in the Exercises for Section 7.3.4.2 and the Exercise for Section 7.3.4.3 are included in this case,
b. Compare the results with the predictions of the maximum stress criterion in the Exercise for Section 7.3.1.4.

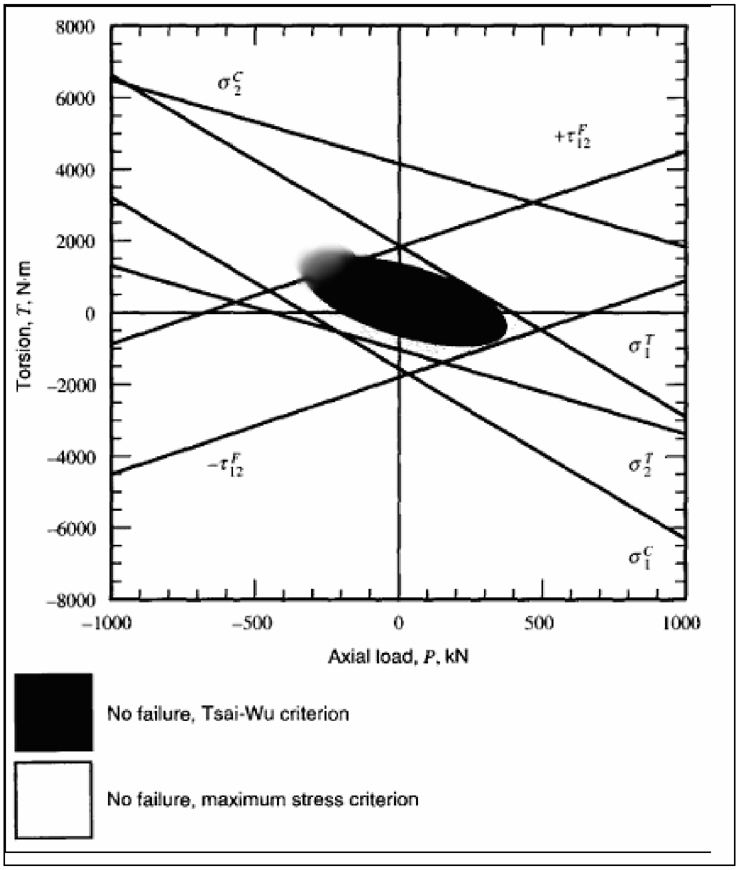


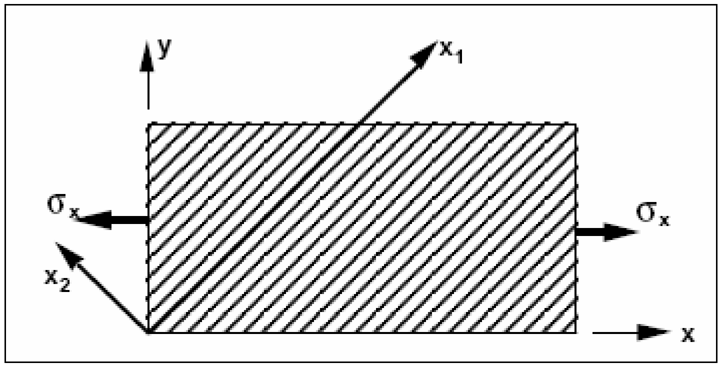
Figure M7.3.4.18 Comparison between maximum stress criterion and Tsai-Wu criterion for +20° layers for the tube subjected to combined axial load P and torsion T

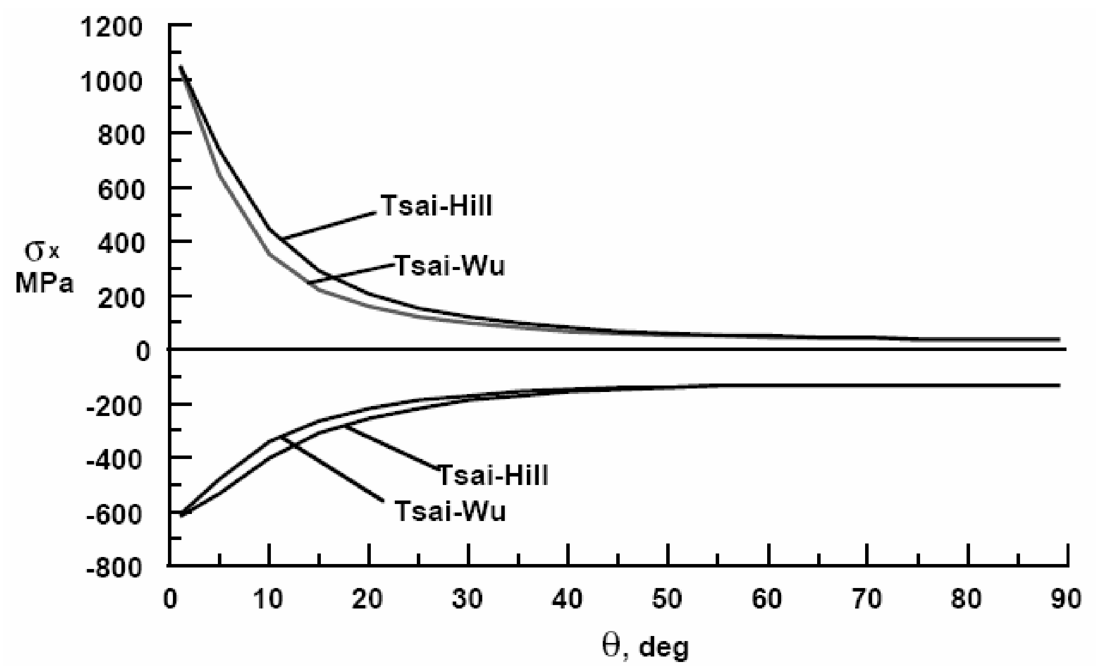
Summary

This completes our chapters on failure. Two distinctly different failure criteria have been discussed and it has been shown in examples that they can lead to different results. Other failure criteria will, of course, lead to other results. The methodologies and procedures developed with the two criteria presented can be used with other criteria. In fact, besides presenting two criteria that are in use, the value of these past chapters is that they do point the way for performing a failure analysis. Stresses must be calculated, these stresses must be used in the equations representing the failure criteria, and the results interpreted. This approach would be followed with any stress-based criterion. With a strain-based criterion, strains in the principal material system would be used rather than stresses. The key issue with any one failure criterion is, "Does it accurately predict failure for your problem?" Generally, considerable testing is required to determine if this is the case. Unfortunately, there does not appear to be one universal criterion which works well for all situations and all materials. Material properties, lamination sequence, and type of loading all seem to influence which criterion works the best. For each particular class of problems and class of materials, a careful study of test data and predictions must be conducted before generalizations can be made. We suggest that more than one criterion be used and the results compared, as we have done here. Competing views are helpful!

M7.4 Comparison of Failure Theories

7.4.1 Uniaxial Strength of Off-AxiLamina Tsai-Hill & Tsai-Wu Theories





Theory	Physical basis	Operational convenience	Required operational convenience
Maximum stress	Tensile behaviour of brittle material	Inconvenient	Few parameters by simple testing
Maximum strain	Tensile behaviour of brittle material Some stress interaction	Inconvenient	Few parameters by simple testing
Deviatoric strain energy (Tsai-Hill)	Ductile behavior of anisotropic materials "Curve fitting" for heterogeneous brittle composites	Can be programmed Different functions required for tensile and compressive strengths	Biaxial testing is needed in addition to uniaxial testing
Interactive tensor polynomial	Mathematically consistent Reliable "curve fitting"	General and comprehensive; operationally simple	Numerous parameters Comprehensive experimental program needed

Table M7.4.1 Comparison of Failure Theories